12. Higher Order Derivatives

Exercise 12.1

26. Question

If y = $\tan^{-1} x$, show that $(1 + x^2) \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$.

Answer

Formula: –

(i) $\frac{dy}{dx} = y_1$ and $\frac{d^2y}{dx^2} = y_2$

(ii) $\frac{d(\tan^{-1}x)}{dx} = \frac{1}{1 + x^2}$

(iii) $\frac{d}{dx}x^n = nx^{n-1}$

(iv) chain rule $\frac{df}{dx} = \frac{d(wou)}{dt}.\frac{dt}{dx} = \frac{dw}{ds}.\frac{ds}{dt}.\frac{dt}{dx}$

Given: –

Y = $\tan^{-1}x$

Differentiating w.r.t x

$\frac{dy}{dx} = \frac{d(\tan^{-1}x)}{dx}$

Using formula(ii)

$\Rightarrow \frac{dy}{dx} = \frac{1}{1 + x^2}$

$\Rightarrow (1 + x^2)\frac{dy}{dx} = 1$

Again Differentiating w.r.t x

Using formula(iii)

$(1 + x^2)\frac{dy}{dx} + 2x\frac{dy}{dx} = 0$

Hence proved.

27. Question

If y = {log (x + $\sqrt{x^2 + 1})^2$, show that $(1 + x^2) \frac{d^2y}{dx^2} + x\frac{dy}{dx} = 2$.

Answer

Formula: –

(i) $\frac{dy}{dx} = y_1$ and $\frac{d^2y}{dx^2} = y_2$

(ii) $\frac{d(\log x)}{dx} = \frac{1}{x}$

(iii) $\frac{d}{dx}x^n = nx^{n-1}$

(iv) chain rule $\frac{df}{dx} = \frac{d(wou)}{dt}.\frac{dt}{dx} = \frac{dw}{ds}.\frac{ds}{dt}.\frac{dt}{dx}$

Given: –

$y = \left[\log\left(x + \sqrt{1 + x^2}\right)\right\}^2$

Differentiating w.r.t x

$\frac{dy}{dx} = \frac{d[\log(x + \sqrt{1 + x^2})\}^2}{dx}$

Using formula(ii)

$\Rightarrow \frac{dy}{dx} = 2\log\left(x + \sqrt{1 + x^2}\right).\frac{1}{\left(x + \sqrt{1 + x^2}\right)}.\left(1 + \frac{2x}{2\sqrt{1 + x^2}}\right)$

Using formula(i)

$\Rightarrow y_1 = \frac{2\log\left(x + \sqrt{1 + x^2}\right)}{x + \sqrt{1 + x^2}}.\frac{x + \sqrt{1 + x^2}}{\sqrt{1 + x^2}}$

$\Rightarrow y_1 = \frac{2\log\left(x + \sqrt{1 + x^2}\right)}{\sqrt{1 + x^2}}$

Squaring both sides

$(y_1)^2 = \frac{4}{1 + x^2}[\log\left(x + \sqrt{1 + x^2}\right)$

Differentiating w.r.t x

$\Rightarrow (1 + x^2)y_2y_1 + 2x(y_1)^2 = 4y_1$

Using formual(iii)

$\Rightarrow (1 + x^2)y_2 + xy_1 = 2$

Hence proved

28. Question

If y = $(\tan^{-1} x)^2$, then prove that $(1 - x2)^2\ y_2 + 2x\ (1 + x^2)\ y_1 = 2$

Answer

Formula: –

(i) $\frac{dy}{dx} = y_1$ and $\frac{d^2y}{dx^2} = y_2$

(ii) $\frac{d(\tan^{-1}x)}{dx} = \frac{1}{1 + x^2}$

(iii) $\frac{d}{dx}x^n = nx^{n-1}$

Given: –

$Y = (\tan^{-1}x)^2$

Then

$$\frac{dy}{dx} = \frac{d(\tan^{-1}x)^2}{dx}$$

Using formula (ii)&(i)

$$y_1 = 2\tan^{-1}x\frac{dy}{dx}(\tan^{-1}x)$$

$$\Rightarrow y_1 = 2\tan^{-1}x.\frac{1}{1 + x^2}$$

Again differentiating with respect to x on both the sides,we obtain

$(1 + x^2)y_2 + 2xy_1 = 2\left(\frac{1}{1+x^2}\right)$ using formula(i)&(iii)

$$\Rightarrow (1 + x^2)^2y_2 + 2x(1 + x^2)y_1 = 2$$

Hence proved.

29. Question

If y = cot x show that $\frac{d^2y}{dx^2} + 2y\frac{dy}{dx} = 0$.

Answer

Formula: –

(i) $\frac{dy}{dx} = y_1$ and $\frac{d^2y}{dx^2} = y_2$

(ii) $\frac{d(\cot x)}{dx} = -\text{cosec}^2x$

(iii) $\frac{d}{dx}x^n = nx^{n-1}$

(iv) chain rule $\frac{df}{dx} = \frac{d(wou)}{dt}.\frac{dt}{dx} = \frac{dw}{ds}.\frac{ds}{dt}.\frac{dt}{dx}$

Given: –

Y = cotx

Differentiating w.r.t. x

$$\frac{dy}{dx} = \frac{d(\cot x)}{dx}$$

Using formula (ii)

$$\Rightarrow \frac{dy}{dx} = -\text{cosec}^2x$$

Differentiating w.r.t x

$$\frac{d^2y}{dx^2} = -[2\text{cosec}x(-\text{cosec}x\cot x)]$$

Using formual (iii)

$$\Rightarrow \frac{d^2y}{dx^2} = 2\text{cosec}^2x\cot x$$

$$\Rightarrow \frac{d^2y}{dx^2} = -2\frac{dy}{dx}.y$$

$$\Rightarrow \frac{d^2y}{dx^2} + 2y\frac{dy}{dx} = 0$$

Hence proved.

30. Question

Find $\frac{d^2y}{dx^2}$, where $y = \log\left(\frac{x^2}{e^2}\right)$.

Answer

Formula: –

(i) $\frac{dy}{dx} = y_1$ and $\frac{d^2y}{dx^2} = y_2$

(ii) $\frac{d(e^{ax})}{dx} = ae^{ax}$

(iii) $\frac{d}{dx}x^n = nx^{n-1}$

Given: –

$$y = \log\left(\frac{x^2}{e^2}\right)$$

Differentiating w.r.t x

$$\frac{dy}{dx} = \frac{1}{\frac{x^2}{e^2}} \cdot \frac{1}{e^2} 2x = \frac{2}{x}$$

Again Differentiating w.r.t x

$$\frac{d^2y}{dx^2} = 2\left(-\frac{1}{x^2}\right) = -\frac{2}{x^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-2}{x^2}$$

31. Question

If y = e^x(sin x + cos x) prove that $\frac{d^2y}{dx^2} - 1\frac{dy}{dx} + 2y = 0$.

Answer

Formula: –

(i) $\frac{dy}{dx} = y_1$ and $\frac{d^2y}{dx^2} = y_2$

(ii) $\frac{d(e^{ax})}{dx} = ae^{ax}$

(iii) $\frac{d}{dx}x^n = nx^{n-1}$

(iv) chain rule $\frac{df}{dx} = \frac{d(wou)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$

Given: –

$y = ae^{2x} + be^{-2x}$

Differentiating w.r.t x

$$\frac{dy}{dx} = 2ae^{2x} + be^{(-x)}(-1)$$

$$\Rightarrow \frac{dy}{dx} = 2ae^{2x} - be^{-x}$$

Differentiating w.r.t x

$$\frac{d^2y}{dx^2} = 2ae^{2x}(2) - be^{-x}(-1)$$

$$\Rightarrow \frac{d^2y}{dx^2} = 4ae^{2x} + be^{-x}$$

Adding and subtracting be^{-x} on RHS

$$\frac{d^2y}{dx^2} = 4ae^{2x} + 2be^{-x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2(ae^{2x} + be^{-x}) + 2ae^{2x} - be^{-x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2y + \frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$$

32. Question

If y = e^x (sin x + cos x) Prove that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

Answer

Formula: –

(i) $\frac{dy}{dx} = y_1$ and $\frac{d^2y}{dx^2} = y_2$

(ii) $\frac{d(e^{ax})}{dx} = ae^{ax}$

(iii) $\frac{d}{dx}x^n = nx^{n-1}$

(iv) chain rule $\frac{df}{dx} = \frac{d(wou)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$

Given: –

$y = e^x(\sin x + \cos x)$

differentiating w.r.t x

$$\frac{dy}{dx} = e^x(\cos x - \sin x) + (\sin x + \cos x)e^x$$

$$\Rightarrow \frac{dy}{dx} = y + e^x(\cos x - \sin x)$$

Differentiating w.r.t x

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x(-\sin x - \cos x) + (\cos x - \sin x)e^x$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy}{dx} - y + (\cos x - \sin x)e^x$$

Adding and subtracting y on RHS

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} - y + (\cos x - \sin x)e^x + y - y$$

$$\Rightarrow \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

Hence proved

33. Question

If $y = \cos^{-1} x$, find $\frac{d^2y}{dx^2}$ in terms of y alone.

Answer

Formula: –

$$\text{(i)} \frac{dy}{dx} = y_1 \text{ and} \frac{d^2y}{dx^2} = y_2$$

$$\text{(ii)} \frac{d(\cos^{-1}x)}{dx} = \frac{-1}{\sqrt{1 + x^2}}$$

$$\text{(iii) chain rule } \frac{df}{dx} = \frac{d(wou)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

Given: – $y = \cos^{-1}x$

Then,

$$\frac{dy}{dx} = \frac{d(\cos^{-1}x)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1 + x^2}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d\left[-\left(\sqrt{1 + x^2}\,\right)\right]^{-1}}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-1.\,(1 - x^2)^{-\frac{3}{2}}}{2} \cdot \frac{d(1 - x^2)}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{2\sqrt{(1 - x^2)^3}} . (-2x)$$

$$\frac{d^2y}{dx^2} = \frac{-x}{\sqrt{(1 - x^2)^3}} \dots\dots (i)$$

$y = \cos^{-1} x$

$\Rightarrow x = \cos y$

Putting $x = \cos y$ in equation(i), we obtain

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-\cos y}{\sqrt{(1-\cos^2 y)^3}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-\cos y}{\sqrt{(\sin^2 y)^3}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-\cos y}{\sin^3 y}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-\cos y}{\sin y}.\frac{1}{\sin^2 y}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\cot y.\mathrm{cosec}^2 y$$

34. Question

If $y = e^{a\cos^{-1}x}$, prove that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$

Answer

Formula: –

(i) $\frac{dy}{dx} = y_1$ and $\frac{d^2y}{dx^2} = y_2$

(ii) $\frac{d(\log x)}{dx} = \frac{1}{x}$

(ii) $\frac{d(\cos^{-1}x)}{dx} = \frac{-1}{\sqrt{1 + x^2}}$

(iii) $\frac{d}{dx}x^n = nx^{n-1}$

(iv) chain rule $\frac{df}{dx} = \frac{d(wou)}{dt}.\frac{dt}{dx} = \frac{dw}{ds}.\frac{ds}{dt}.\frac{dt}{dx}$

(v) logarithms differentiation $\frac{dy}{dx} = y\left[\frac{v(x)}{u(x)}.u'(x) + v'(x).\log[u(x)]\right]$

Given: –

$y = e^{a\cos^{-1}x}$

Taking logarithm on both sides we obtain

$$\frac{1}{y}\frac{dy}{dx} = a\frac{-1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{-ay}{\sqrt{1-x^2}}$$

By squaring both sides, wee obtain

$$\left(\frac{dy}{dx}\right)^2 = \frac{a^2y^2}{1-x^2}$$

$$\Rightarrow (1-x^2).\left(\frac{dy}{dx}\right)^2 = a^2y^2$$

$$\Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = a^2y^2$$

Again differentiating both sides with respect to x,we obtain

$$\left(\frac{dy}{dx}\right)^2 \cdot \frac{d(1-x^2) + (1-x^2)}{dx} \cdot \frac{d}{dx}\left[\left(\frac{dy}{dx}\right)^2\right] = a^2\frac{d(y^2)}{dx}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 [(-2x) + (1-x^2)]2.\frac{dy}{dx}.\frac{d^2 y}{dx^2} = a^2.2y.\frac{dy}{dx}$$

$$\Rightarrow -x\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2} = a^2y$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$$

Hence proved

35. Question

If $y = 500\,e^{7x} + 600\,e^{-7x}$, show that $\frac{d^2y}{dx^2} = 49y$.

Answer

Formula: –

$$(i)\,\frac{dy}{dx} = y_1 \text{ and} \frac{d^2y}{dx^2} = y_2$$

$$(ii)\,\frac{d(e^{ax})}{dx} = ae^{ax}$$

$$(iii)\,\frac{d}{dx}x^n = nx^{n-1}$$

$$(iv)\text{ chain rule } \frac{df}{dx} = \frac{d(wou)}{dt}.\frac{dt}{dx} = \frac{dw}{ds}.\frac{ds}{dt}.\frac{dt}{dx}$$

Given: –

$y = 500e^{7x} + 600e^{-7x}$

$$\frac{dy}{dx} = 500.\frac{d(e^{7x})}{dx} + 600.\frac{d(e^{-7x})}{dx}$$

$$\Rightarrow \frac{dy}{dx} = 500e^{7x}.\frac{d(7x)}{dx} + 600.e^{7x}.\frac{d(-7x)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = 3500e^{7x} - 4200e^{-7x}$$

$$\Rightarrow \frac{dy}{dx} = 49(500e^{7x} + 600e^{-7x})$$

$$\Rightarrow \frac{dy}{dx} = 49y$$

Hence proved.

36. Question

If x = 2 cos t – cos 2t, y = 2 sin t – sin 2t, find $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{2}$.

Answer

Formula: –

(i) $\frac{dy}{dx} = y_1$ and $\frac{d^2y}{dx^2} = y_2$

(ii) $\frac{d}{dx}\cos x = \sin x$

(iii) $\frac{d}{dx}\sin x = -\cos x$

(iv) $\frac{d}{dx}x^n = nx^{n-1}$

(v) chain rule $\frac{df}{dx} = \frac{d(wou)}{dt}.\frac{dt}{dx} = \frac{dw}{ds}.\frac{ds}{dt}.\frac{dt}{dx}$

(vi)parameteric forms $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Given: –

x = 2cost – cos2t

y = 2sint – sin2t

differentiating w.r.t t

$\frac{dy}{dx} = 2(-\sin t) - 2(-\sin 2t)$

$\Rightarrow \frac{dy}{dt} = 2\cos t - 2\cos 2t$

Dividing both

$\frac{dy}{dx} = \frac{2(\cos t - \cos 2t)}{2(\sin 2t - \sin t)}$

Differentiating w.r.t t

$\Rightarrow \frac{d\frac{dy}{dx}}{dt} = \frac{(\sin 2t - \sin t)(-\sin t + 2\sin 2t) - (\cos t - \cos 2t)(2\cos 2t - \cos t)}{(\sin 2t - \sin t)^2}$

Dividing

$\frac{d^2y}{dx^2} = \frac{(\sin 2t - \sin t)(2\sin t - \sin t) - (\cos t - \cos 2t)(2\cos 2t - \cos t)}{2(\sin 2t - \sin t)^3}$

Putting $t = \frac{\pi}{2}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{1 + 2}{-2} = -\frac{3}{2}$

37. Question

If x = 4z² + 5, y = 6z² + 7z + 3, find $\frac{d^2y}{dx^2}$.

Answer

Formula: –

(i) $\frac{dy}{dx} = y_1$ and $\frac{d^2y}{dx^2} = y_2$

(ii) $\frac{d}{dx}x^n = nx^{n-1}$

(iii) chain rule $\frac{df}{dx} = \frac{d(wou)}{dt}.\frac{dt}{dx} = \frac{dw}{ds}.\frac{ds}{dt}.\frac{dt}{dx}$

(iv)parameteric forms $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Given: –

$x = 4z^2 + 5, y = 6z^2 + 72 + 3$

Differentiating both w.r.t z

$\frac{dx}{dz} = 8z + 0$

$\Rightarrow \frac{dx}{dz} = \frac{12z + 7}{8z}$

and$\Rightarrow \frac{dy}{dz} = 12z + 7$

differentiating w.r.t z

$\frac{d\left(\frac{dy}{dx}\right)}{dz} = 0 + \frac{7}{8}\left(\frac{-1}{z^2}\right)$

Dividing

$\Rightarrow \frac{d^2y}{dx^2} = \frac{-7}{8z^2 \times 8z} = \frac{-7}{64z^3}$

38. Question

If y = log (1 + cos x), prove that $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2}.\frac{dy}{dx} = 0$

Answer

Formula: –

(i) $\frac{dy}{dx} = y_1$ and $\frac{d^2y}{dx^2} = y_2$

(ii) $\frac{d}{dx}\cos x = \sin x$

(iii) $\frac{d}{dx}\sin x = -\cos x$

(iv) $\frac{d}{dx}x^n = nx^{n-1}$

(v) chain rule $\frac{df}{dx} = \frac{d(wou)}{dt}.\frac{dt}{dx} = \frac{dw}{ds}.\frac{ds}{dt}.\frac{dt}{dx}$

Given: –

Y = log(1 + cosx)

Differentiating w.r.t x

$$\frac{dy}{dx} = \frac{1}{1 + \cos x}.(-\sin x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\sin x}{1 + \cos x}$$

Differentiating w.r.t.x

$$\frac{d^2 y}{dx^2} = -\left[\frac{(1 + \cos x)\cos x - \sin x(-\sin x)}{(1 + \cos x)^2}\right]$$

$$\Rightarrow \frac{d^2 y}{dx^2} = -\left[\frac{(\cos x) + \cos^2 x + \sin^2 x}{(1 + \cos x)^2}\right]$$

$$\Rightarrow \frac{d^2 y}{dx^2} = -\left[\frac{1 + \cos x}{(1 + \cos x)^2}\right]$$

$$\Rightarrow \frac{d^2 y}{dx^2} = -\frac{1}{1 + \cos x}$$

Differentiating w.r.t x

$$\frac{d^3y}{dx^3} = -\left(\frac{1}{(1 + \cos x)^2} \times -\sin x\right)$$

$$\Rightarrow \frac{d^3y}{dx^3} = -\left(\frac{-\sin x}{1 + \cos x}\right) \times \left(\frac{-1}{1 + \cos x}\right)$$

$$\Rightarrow \frac{d^3y}{dx^3} = -\frac{dy}{dx}.\frac{d^2y}{dx^2}$$

$$\Rightarrow \frac{d^3y}{dx^3} + \frac{dy}{dx}.\frac{d^2y}{dx^2} = 0$$

39. Question

If y = sin (log x), prove that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

Answer

Formula: –

(i) $\frac{dy}{dx} = y_1$ and $\frac{d^2y}{dx^2} = y_2$

(ii) $\frac{d(\log x)}{dx} = \frac{1}{x}$

(iii) $\frac{d}{dx}\cos x = \sin x$

(iv) $\frac{d}{dx}\sin x = -\cos x$

(v) $\frac{d}{dx}x^n = nx^{n-1}$

(vi) chain rule $\frac{df}{dx} = \frac{d(wou)}{dt}.\frac{dt}{dx} = \frac{dw}{ds}.\frac{ds}{dt}.\frac{dt}{dx}$

Given: –

y = sin(logx)

$\frac{dy}{dx} = \cos(\log x)\frac{1}{x}$

$\Rightarrow x\frac{dy}{dx} = \cos(\log x)$

$\Rightarrow x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -\sin(\log x)\frac{1}{x}$

$\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y$

$\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

Hence proved.

40. Question

If $y = 3e^{2x} + 2e^{3x}$, prove that $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$.

Answer

Formula: –

(i) $\frac{dy}{dx} = y_1$ and $\frac{d^2y}{dx^2} = y_2$

(ii) $\frac{d(e^{ax})}{dx} = ae^{ax}$

(iii) $\frac{d}{dx}x^n = nx^{n-1}$

Given: –

$y = 3e^{2x} + 2e^{3x}$

$\Rightarrow \frac{dy}{dx} = 6e^{2x} + 6e^{3x}$

$\Rightarrow \frac{d^2y}{dx^2} = 12e^{2x} + 18e^{3x}$

Hence

$$\Rightarrow \frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 6(2e^{2x} + 3e^{3x}) - 30(e^{2x} + e^{3x}) + 6(3e^{2x} + 2e^{3x})$$
$$= 0$$

41. Question

If $y = (\cot^{-1} x)^2$, prove that $y_2 (x^2 + 1)^2 + 2x(x^2 + 1)y_1 = 2$.

Answer

(i) $\frac{dy}{dx} = y_1$ and $\frac{d^2y}{dx^2} = y_2$

(ii) $\frac{d}{dx}\cot^{-1}x = \frac{-1}{1 + x^2}$

(iii) $\frac{d}{dx}x^n = nx^{n-1}$

(iv) chain rule $\frac{df}{dx} = \frac{d(wou)}{dt}.\frac{dt}{dx} = \frac{dw}{ds}.\frac{ds}{dt}.\frac{dt}{dx}$

Given: –

$y = (\cot^{-1}x)^2$

differentiating w.r.t x

$\frac{dy}{dx} = y_1 = 2\cot^{-1}x.\left[\frac{-1}{1 + x^2}\right]$

$\Rightarrow y_1 = \frac{-2\cot^{-1}x}{1 + x^2}$

Differentiating w.r.t x

$\Rightarrow (1 + x^2)y_2 + 2xy_1 = 2\left(\frac{1}{1 + x^2}\right)$

$\Rightarrow (1 + x^2)^2 y_2 + 2x(1 + x^2)y_1 = 2$

Hence proved

42. Question

If $y = \operatorname{cosec}^{-1}x, x > 1$, then show that $x(x^2 - 1)\frac{d^2y}{dx^2} + (2x^2 - 1)\frac{dy}{dx} = 0$

Answer

Formula: –

(i) $\frac{dy}{dx} = y_1$ and $\frac{d^2y}{dx^2} = y_2$

(ii) $\frac{d(\operatorname{cosec}^{-1}x)}{dx} = \frac{-1}{|x|\sqrt{x^2 - 1}}$

(iii) $\frac{d}{dx}x^n = nx^{n-1}$

(iv) chain rule $\frac{df}{dx} = \frac{d(wou)}{dt}.\frac{dt}{dx} = \frac{dw}{ds}.\frac{ds}{dt}.\frac{dt}{dx}$

Given: –

$Y = \operatorname{cosec}^{-1}x$

We know that

$$\frac{d(\operatorname{cosec}^{-1}x)}{dx} = \frac{-1}{|x|\sqrt{x^2-1}}$$

Let y = cosec $^{-1}$x

$$\frac{dy}{dx} = \frac{-1}{|x|\sqrt{x^2-1}}$$

Since x>1,|x| = x

$$\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-1}}$$

Differentiating the above function with respect to x

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{x\frac{2x}{2\sqrt{x^2-1}} + \sqrt{x^2-1}}{x^2(x^2-1)}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\frac{x^2}{\sqrt{x^2-1}} + \sqrt{x^2-1}}{x^2(x^2-1)}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{x^2 + x^2 - 1}{x^2(x^2-1)^{\frac{3}{2}}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2x^2-1}{x^2(x^2-1)^{\frac{3}{2}}}$$

Thus

$$x(x^2-1)\frac{d^2y}{dx^2} = \frac{2x^2-1}{x\sqrt{x^2-1}} \ldots\ldots(2)$$

Similarly

$$\Rightarrow [2x^2-1]\frac{dy}{dx} = \frac{-2x^2+1}{x\sqrt{x^2-1}}$$

$$\Rightarrow x(x^2-1)\frac{d^2y}{dx^2} + [2x^2-1]\frac{dy}{dx} = \frac{2x^2-1}{x\sqrt{x^2-1}} + \frac{-2x^2+1}{x\sqrt{x^2-1}} = 0$$

Hence proved.

43. Question

If $x = \cos t + \log \tan\frac{t}{2}, y = \sin t$, then find the value of $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$.

Answer

Formula: –

(i) $\frac{dy}{dx} = y_1$ and $\frac{d^2y}{dx^2} = y_2$

(ii) $\frac{d}{dx}\cos x = \sin x$

(iii) $\frac{d}{dx}\sin x = -\cos x$

(iv) $\frac{d}{dx}\log x = \frac{1}{x}$

(v) $\frac{d}{dx}\tan x = \sec^2 x$

(vi) $\frac{d}{dx}x^n = nx^{n-1}$

(v) chain rule $\frac{df}{dx} = \frac{d(wou)}{dt}.\frac{dt}{dx} = \frac{dw}{ds}.\frac{ds}{dt}.\frac{dt}{dx}$

(vi)parameteric forms $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Given: –

$x = \cos t + \log\tan\frac{t}{2}, y = \sin t$

Differentiating with respect to t ,we have

$$\frac{dx}{dt} = -\sin t + \frac{1}{\tan\frac{t}{2}} \times \sec^2\left(\frac{t}{2}\right) \times \frac{1}{2}$$

$$\Rightarrow \frac{dx}{dt} = -\sin t + \frac{1}{\frac{\sin\left(\frac{t}{2}\right)}{\cos\left(\frac{t}{2}\right)}} \times \frac{1}{\cos^2\frac{t}{2}} \times \frac{1}{2}$$

$$\Rightarrow \frac{dx}{dt} = -\sin t + \frac{1}{2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)}$$

$$\Rightarrow \frac{dx}{dt} = -\sin t + \frac{1}{\sin t}$$

$$\Rightarrow \frac{dx}{dt} = \frac{1-\sin^2 t}{\sin t}$$

$$\Rightarrow \frac{dx}{dt} = \frac{\cos^2 t}{\sin t}$$

$$\Rightarrow \frac{dx}{dt} = \cos t.\cot t$$

Now find the value of $\frac{dy}{dt}$

$$\frac{dy}{dt} = \cos t$$

Now

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \cos t \times \frac{1}{\cos t.\cot t}$$

$$\Rightarrow \frac{dy}{dx} = \tan t$$

We have

$\frac{dy}{dt} = \cos t$

Differentiating with w.r.t t

$\frac{d^2y}{dt^2} = -\sin t$

At $t = \frac{\pi}{4}$

$$\left(\frac{d^2y}{dt^2}\right)_{t=\frac{\pi}{4}} = -\sin\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\tan t)}{\cos t.\cot t}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\sec^2 t}{\cos t.\cot t}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\sec^2 t}{\cos^2 t}.\sin t$$

$$\Rightarrow \frac{d^2y}{dx^2} = \sec^4 t \times \sin t$$

Now putting $t = \frac{\pi}{4}$

$$\left(\frac{d^2y}{dx^2}\right)_{t=\frac{\pi}{4}} = \sec^4\frac{\pi}{4}.\sin\left(\frac{\pi}{4}\right) = 2$$

44. Question

If x = a sin t and $y = a\left(\cos t + \log \tan\frac{t}{2}\right)$, find $\frac{d^2y}{dx^2}$.

Answer

Formula: –

(i) $\frac{dy}{dx} = y_1$ and $\frac{d^2y}{dx^2} = y_2$

(ii) $\frac{d}{dx}\cos x = \sin x$

(iii) $\frac{d}{dx}\sin x = -\cos x$

(iv) $\frac{d}{dx}x^n = nx^{n-1}$

(v) chain rule $\frac{df}{dx} = \frac{d(wou)}{dt}.\frac{dt}{dx} = \frac{dw}{ds}.\frac{ds}{dt}.\frac{dt}{dx}$

(vi)parameteric forms $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Given: –

$x = \text{atsint and } y = a\left(\text{cost} + \log\tan\left(\frac{t}{2}\right)\right)$

$\frac{dx}{dt} = \text{acost}$

$\Rightarrow \frac{d^2y}{dt^2} = -\text{asint}$

$\Rightarrow \frac{dy}{dt} = -\text{asint} + \frac{a}{\tan\left(\frac{t}{2}\right)} \times \sec^2\frac{t}{2} \times \frac{1}{2}$

$\Rightarrow \frac{dy}{dt} = -\text{asint} + \frac{a}{2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)}$

$\Rightarrow \frac{dy}{dt} = -\text{asint} + \text{acosect}$

$\Rightarrow \frac{d^2y}{dt^2} = -\text{acost} - \text{acosectcott}$

$\frac{d^2y}{dx^2} = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{\text{acost}(-\text{acost} - \text{acosectcott}) - (-\text{asint} + \text{acosect})(-\text{asint})}{(\text{acost})^3}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{-a^2(\cos^2 t + \sin^2 t) - a^2\cot^2 t + a^2}{a^3\cos^3 t}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{\text{asin}^2 t\,\text{cost}}$

45. Question

If x = a (cos t + t sin t) and y = a (sin t – t cos t), then find the value of $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$

Answer

Formula: –

(i) $\frac{dy}{dx} = y_1$ and $\frac{d^2y}{dx^2} = y_2$

(ii) $\frac{d}{dx}\text{cosx} = \text{sinx}$

(iii) $\frac{d}{dx}\text{sinx} = -\text{cosx}$

(iv) $\frac{d}{dx}x^n = nx^{n-1}$

(v) chain rule $\frac{df}{dx} = \frac{d(wou)}{dt}.\frac{dt}{dx} = \frac{dw}{ds}.\frac{ds}{dt}.\frac{dt}{dx}$

(vi)parameteric forms $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Given: -

$x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$

$\frac{dy}{dt} = a\cos t - a\cos t + at\sin t = at\sin t$

$\Rightarrow \frac{d^2y}{dt^2} = at\cos t + a\sin t$

$\Rightarrow \frac{dx}{dt} = -a\sin t + at\cos t + a\sin t = at\cos t$

$\Rightarrow \frac{d^2x}{dt^2} = -at\sin t + a\cos t$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{at\cos t(at\cos t + a\sin t) - (-at\sin t + a\cos t)(at\sin t)}{(a\cos t)^3}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{(at\cos t)^3}$

Putting $t = \frac{\pi}{4}$

$\left(\frac{d^2y}{dx^2}\right)_{t=\frac{\pi}{4}} = \frac{1}{a\cos^3\frac{\pi}{4}.a\frac{\pi}{4}} = \frac{8\sqrt{2}}{\pi a}$

46. Question

If $x = a\left(\cos t + \log \tan\frac{t}{2}\right)$, y = a sin t, evaluate $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{3}$

Answer

Formula: -

(i) $\frac{dy}{dx} = y_1$ and $\frac{d^2y}{dx^2} = y_2$

(ii) $\frac{d}{dx}\cos x = \sin x$

(iii) $\frac{d}{dx}\sin x = -\cos x$

(iv) $\frac{d}{dx}x^n = nx^{n-1}$

(v) chain rule $\frac{df}{dx} = \frac{d(wou)}{dt}.\frac{dt}{dx} = \frac{dw}{ds}.\frac{ds}{dt}.\frac{dt}{dx}$

(vi)parameteric forms $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Given: –

$x = a(\cos t + \log\tan\frac{t}{2}), y = \sin t$

Differentiating with respect to t ,we have

$$\Rightarrow \frac{dx}{dt} = -a\sin t + a\frac{1}{\tan\frac{t}{2}} \times \sec^2\left(\frac{t}{2}\right) \times \frac{1}{2}$$

$$\Rightarrow \frac{dx}{dt} = -a\sin t + a\frac{1}{\frac{\sin\left(\frac{t}{2}\right)}{\cos\left(\frac{t}{2}\right)}} \times \frac{1}{\cos^2\frac{t}{2}} \times \frac{1}{2}$$

$$\Rightarrow \frac{dx}{dt} = -a\sin t + a\frac{1}{2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)}$$

$$\Rightarrow \frac{dx}{dt} = -a\sin t + a\frac{1}{\sin t} = -a\sin t + a\operatorname{cosec} t$$

Now find the value of $\frac{dy}{dt}$

$$\frac{dy}{dt} = a\cos t$$

$$\Rightarrow \frac{d^2y}{dt^2} = -a\sin t$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-a\sin t(-a\sin t + a\operatorname{cosec} t) — (-a\cos t - a\operatorname{cosec} t\cot t)(-a\cos t)}{(a\operatorname{cosec} t - a\sin t)^3}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{a^2(\cos^2 t + \sin^2 t) + a^2\cot^2 t - a^2}{(a\operatorname{cosec} t - a\sin t)^3}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\sin t}{a\cos^4 t}$$

$$\left(\frac{d^2y}{dx^2}\right)_{t=\frac{\pi}{3}} = \frac{\sin\frac{\pi}{3}}{a\cos^4\frac{\pi}{3}} = \frac{8\sqrt{3}}{a}$$

47. Question

If x = a (cos 2t + 2t sin 2t) and y = a (sin 2t – 2t cos 2t), then find $\frac{d^2y}{dx^2}$.

Answer

Formula: –

(i) $\frac{dy}{dx} = y_1$ and $\frac{d^2y}{dx^2} = y_2$

(ii) $\frac{d}{dx}\cos x = \sin x$

(iii) $\frac{d}{dx}\sin x = -\cos x$

(iv) $\frac{d}{dx}x^n = nx^{n-1}$

(v) chain rule $\frac{df}{dx} = \frac{d(wou)}{dt}.\frac{dt}{dx} = \frac{dw}{ds}.\frac{ds}{dt}.\frac{dt}{dx}$

(vi)parametric forms $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Given: –

x = a (cos 2t + 2t sin 2t)

$\Rightarrow \frac{dx}{dt} = -2a\sin 2t + 2a\sin 2t + 4atco2t = 4at\cos 2t$

and y = a (sin 2t – 2t cos 2t)

$\Rightarrow \frac{dy}{dt} = 2a\cos 2t - 2a\cos 2t + 4at\sin 2t = 4at\sin 2t$

$\Rightarrow \frac{dy}{dx} = \frac{dt}{dx} \times \frac{dy}{dt} = \frac{\sin 2t}{\cos 2t} = \tan 2t$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{d(\tan 2t)}{dx}$

$\Rightarrow \frac{d^2y}{dx^2} = \sec^2 2t\frac{d(2t)}{dx}$

$\Rightarrow \frac{d^2y}{dx^2} = 2\sec^2 2t\frac{d(t)}{dx}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{\sec^3 2t}{2a}$

48. Question

If x = 3 cot t – 2 $\cos^3$ t, y = 3 sin t – 2 $\sin^3$ t, find $\frac{d^2y}{dx^2}$.

Answer

Formula: –

(i) $\frac{dy}{dx} = y_1$ and $\frac{d^2y}{dx^2} = y_2$

(ii) $\frac{d}{dx}\cos x = \sin x$

(iii) $\frac{d}{dx}\sin x = -\cos x$

(iv) $\frac{d(\cot x)}{dx} = -\mathrm{cosec}^2 x$

(v) $\frac{d}{dx}x^n = nx^{n-1}$

(vi) chain rule $\frac{df}{dx} = \frac{d(wou)}{dt}.\frac{dt}{dx} = \frac{dw}{ds}.\frac{ds}{dt}.\frac{dt}{dx}$

(vii)parameteric forms $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

given: –

x = 3 cot t – 2 $\cos^3$ t, y = 3 sin t – 2 $\sin^3$ t

differentiating both w.r.t t

$$\frac{dx}{dt} = -3\sin t - 6\cos^2 t(-\sin t)$$

$$\frac{dx}{dt} = -3\sin t + 6\cos^2 t\sin t$$

And $y = 3\sin t - 2\sin^3 t$

differentiating both w.r.t t

$$\frac{dy}{dt} = 3\cos t - 6\sin^2 t\cos t$$

Now,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos t - 2\sin^2 t\cos t}{-\sin t + 2\cos^2 t\sin t}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos t[1 - 2\sin^2 t]}{\sin t[2\cos^2 t - 1]}$$

$\Rightarrow \frac{dy}{dx}$ = cot t

differentiating both w.r.t x

$$\frac{d^2 y}{dx^2} = \frac{d(\cot x)}{dx} = -\mathrm{cosec}^2 x$$

49. Question

If x = a sin t – b cos t, y = a cos t + b sin t, prove that $\frac{d^2y}{dx^2} = -\frac{x^2+y^2}{y^3}$.

Answer

Formula: –

(i) $\frac{dy}{dx} = y_1$ and $\frac{d^2y}{dx^2} = y_2$

(ii) $\frac{d}{dx}\cos x = \sin x$

(iii) $\frac{d}{dx}\sin x = -\cos x$

(iv) $\frac{d}{dx}x^n = nx^{n-1}$

(v) chain rule $\frac{df}{dx} = \frac{d(wou)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$

(vi)parameteric forms $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Given: –

x = asint – bcost,y = accost + bsint

differentiating both w.r.t t

$\frac{dx}{dt} = a\cos t + b\sin t, \frac{dy}{dt} = -a\sin t + b\cos t$

$\Rightarrow \frac{dx}{dt} = y, \Rightarrow \frac{dy}{dt} = x$

Dividing both

$\Rightarrow \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = -\frac{x}{y}$

Differentiating w.r.t t

$\Rightarrow \frac{d\left(\frac{dy}{dx}\right)}{dt} = -\frac{y\left(\frac{dx}{dt}\right) - x\left(\frac{dy}{dt}\right)}{y^2}$

Putting the value

$\Rightarrow \frac{d\left(\frac{dy}{dx}\right)}{dt} = -\frac{\{y^2 + x^2\}}{y^2}$

Dividing them

$\Rightarrow \frac{d^2y}{dx^2} = -\frac{\{y^2 + x^2\}}{y^2.y} = -\frac{\{x^2 + y^2\}}{y^3}$

Hence proved.

50. Question

Find A and B so that y = A sin 3x + Bcos 3x satisfies the equation $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 10\cos 3x$.

Answer

Formula: –

(i) $\frac{dy}{dx} = y_1$ and $\frac{d^2y}{dx^2} = y_2$

(ii) $\frac{d}{dx}\cos x = \sin x$

(iii) $\frac{d}{dx}\sin x = -\cos x$

(iv) chain rule $\frac{df}{dx} = \frac{d(wou)}{dt}.\frac{dt}{dx} = \frac{dw}{ds}.\frac{ds}{dt}.\frac{dt}{dx}$

Given: –

y = Asin3x + Bcos3x

differentiating w.r.t x

$\frac{dy}{dx} = 3A\text{xos}3x + 3B(-\sin 3x)$

Again differentiating w.r.t x

$\frac{d^2y}{dx^2} = 3A(-\sin 3x).3 - 3B(\cos 3x).3$

$\Rightarrow \frac{d^2y}{dx^2} = -9(A\sin 3x + B\cos 3x) = -9y$

Now adding

$\frac{d^2y}{dx^2} + \frac{4dy}{dx} + 3y$

$\Rightarrow \frac{d^2y}{dx^2} + \frac{4dy}{dx} + 3y = -9y + 4(3A\cos 3x - 3B\sin 3x) + 3y$

$\Rightarrow \frac{d^2y}{dx^2} + \frac{4dy}{dx} + 3y = 12(A\cos 3x - b\sin 3x) - 6(A\sin 3x + B\cos 3x)$

$\Rightarrow \frac{d^2y}{dx^2} + \frac{4dy}{dx} + 3y = (12A - 6B)\cos 3x - (12B + 6A)\sin 3x$

But given,

$\frac{d^2y}{dx^2} + \frac{4dy}{dx} + 3y = 10\cos 3x$

⇒12A – 6B = 10

⇒ – (12B + 6A) = 0

⇒6A = – 12B

⇒A = – 2B

Puttuing A

$\Rightarrow 12(-2B) - 63 = 10$

$\Rightarrow -24B - 6B = 10$

$\Rightarrow B = -\frac{1}{3}$

$A = -2 \times -\frac{1}{3} = \frac{2}{3}$

And $A = \frac{2}{3}, B = -\frac{1}{3}$

51. Question

If $y = A e^{-kt} \cos(pt + c)$, prove that $\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + n^2y = 0$, where $n^2 = p^2 + k^2$.

Answer

Formula: –

(i) $\frac{dy}{dx} = y_1$ and $\frac{d^2y}{dx^2} = y_2$

(ii) $\frac{d}{dx}e^{ax} = ae^{ax}$

(iii) $\frac{d}{dx}\cos x = \sin x$

(iv) $\frac{d}{dx}\sin x = -\cos x$

(v) chain rule $\frac{df}{dx} = \frac{d(wou)}{dt}.\frac{dt}{dx} = \frac{dw}{ds}.\frac{ds}{dt}.\frac{dt}{dx}$

(vi)parameteric forms $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Given: –

$y = A\,e^{-kt}\cos(pt + c)$

Differentiating w.r.t t

$$\frac{dy}{dt} = A\left(e^{-kt}(-\sin(pt + c).p) + (\cos(pt + c))(-re^{-kt})\right)$$

$$\Rightarrow \frac{dy}{dt} = -Ape^{-kt}(pt + c) - kAe^{-kt}\cos(pt + c)$$

$$\Rightarrow \frac{dy}{dt} = -Ape^{-kt}\sin(pt + c) - ky$$

Differentiating w.r.t t

$$\frac{d^2y}{dt^2} = Apke^{-kt}\sin(pt + c) - p^2y - 2ky_1 + ky_1$$

$$\Rightarrow \frac{d^2y}{dt^2} = Apke^{-kt}\sin(pt + c) - p^2y - 2ky_1 - kApe^{-kt}\sin(pt + c) - k^2y$$

$$\Rightarrow \frac{d^2y}{dt^2} = -(p^2 + k^2)y - 2k\frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dt^2} + 2k\frac{dy}{dx} + n^2y = 0$$

Hence proved

52. Question

If $y = x^n\{a\cos(\log x) + b\sin(\log x)\}$, prove that $x^2\frac{d^2y}{dx^2} + (1-2n)\frac{dy}{dx} + (1+n^2)y = 0$

Answer

Formula: –

(i) $\frac{dy}{dx} = y_1$ and $\frac{d^2y}{dx^2} = y_2$

(ii) $\frac{d}{dx}\cos x = \sin x$

(iii) $\frac{d}{dx}\sin x = -\cos x$

(iv) $\frac{d(\log x)}{dx} = \frac{1}{x}$

(v) $\frac{d}{dx}x^n = nx^{n-1}$

(vi) chain rule $\frac{df}{dx} = \frac{d(wou)}{dt}.\frac{dt}{dx} = \frac{dw}{ds}.\frac{ds}{dt}.\frac{dt}{dx}$

Given: –

$y = x^n(a\cos(\log x) + b\sin(\log x))$

$\Rightarrow y = ax^n\cos(\log x) + bx^n\sin(\log x)$

$$\frac{dy}{dx} = anx^{n-1}\cos(\log x) - ax^{n-1}\sin(\log x) + bx^{n-1}\sin\log x + bx^{n-1}\cos\log x$$

$$\Rightarrow \frac{dy}{dx} = x^{n-1}\cos\log x(na + b) + x^{n-1}\sin(\log x)(bn - a)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(x^{n-1}\cos(\log x)(na + b) + x^{n-1}\sin(\log x)(bn - a)\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = (na + b)[(n-1)x^{n-2}\cos(\log x) - x^{n-2}\sin(\log x)] + (bn - a)[(n-1)x^{n-2}\sin(\log x) + x^{n-2}\cos(\log x)]$$

$$x^2\frac{d^2y}{dx^2} + (1 - 2n)\frac{dy}{dx} + (1 + n^2)y$$

$= x^n (na + b)[(n – 1) \cos(\log x) – \sin (\log x)] + (bn – a) x^n [(n – 1) \sin(\log x) + \cos(\log x)] + (1 – 2n)x^{n-1}\cos(\log x)(na + b) + (1 – 2n)x^{n-1}\sin(\log x)(bn – a) + a(1 + n^2)x^n\cos(\log x) + bx^n(1 + n^2)\sin(\log x)$

$$\Rightarrow x^2\frac{d^2y}{dx^2} + (1 - 2n)\frac{dy}{dx} + (1 + n^2)y = 0$$

53. Question

If $y = a \{x + \sqrt{x^2 + 1}\}^n + b\{x - \sqrt{x^2 + 1}\}^{-n}$, prove that $(x^2 - 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - n^2 = 0$.

Answer

Formula: –

(i) $\frac{dy}{dx} = y_1$ and $\frac{d^2y}{dx^2} = y_2$

(ii) $\frac{d}{dx}x^n = nx^{n-1}$

(iii) chain rule $\frac{df}{dx} = \frac{d(wou)}{dt}.\frac{dt}{dx} = \frac{dw}{ds}.\frac{ds}{dt}.\frac{dt}{dx}$

Given: –

$$y = a\left\{x + \sqrt{x^2 + 1}\right\}^n + b\left\{x - \sqrt{x^2 + 1}\right\}^{-n}$$

$$\frac{dy}{dx} = na\{x + x^2 + 1\}^{n-1}\left[1 + x(x^2 + 1)^{-\frac{1}{2}}\right] - nb\left\{x - \sqrt{x^2 + 1}\right\}^{-n-1}\left[1 - x(x^2 + 1)^{-\frac{1}{2}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{na\{x + x^2 + 1\}^n}{\sqrt{x^2 + 1}} + \frac{nb\{x + x^2 + 1\}^{-n}}{\sqrt{x^2 + 1}}$$

$$\Rightarrow \frac{xdy}{dx} = \frac{nx}{\sqrt{x^2 + 1}}y$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{nx}{\sqrt{x^2 + 1}}\frac{dy}{dx} + y\left[\frac{\sqrt{x^2 + 1} - x^2(x^2 + 1)^{-\frac{1}{2}}}{x^2 + 1}\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{n^2x^2}{x^2 + 1} + y\left[\frac{1}{(x^2 + 1)\sqrt{x^2 + 1}}\right]$$

$$\Rightarrow (x^2 - 1)\frac{d^2y}{dx^2} = \frac{n^2x^4\left(\sqrt{x^2 + 1}\right) + x^2y}{x^2 + 1\sqrt{x^2 + 1}} - \frac{n^2x^2\left(\sqrt{x^2 + 1}\right) + y}{x^2 + 1\left(\sqrt{x^2 + 1}\right)}$$

Now

$$\Rightarrow (x^2 - 1)\frac{d^2y}{dx^2} + \frac{xdy}{dx} - ny = \frac{n^2x^4\left(\sqrt{x^2 + 1}\right) + x^2y}{(x^2 + 1)\sqrt{x^2 + 1}} - \frac{n^2x^2\left(\sqrt{x^2 + 1}\right) + y}{(x^2 + 1)\left(\sqrt{x^2 + 1}\right)} - ny = 0$$

1 A. Question

Find the second order derivatives of each of the following functions:

$x^3 + \tan x$

Answer

<u>Basic idea:</u>

√Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

√The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v, i.e. f = v(u(x)). For the sake of simplicity just assume t = u(x)

Then f = v(t). By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$$

√Product rule of differentiation- $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

√Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

Let's solve now:

Given, $y = x^3 + \tan x$

We have to find $\frac{d^2y}{dx^2}$

As $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So lets first find $\frac{dy}{dx}$ and differentiate it again.

$\therefore \frac{dy}{dx} = \frac{d}{dx}(x^3 + \tan x) = \frac{d}{dx}(x^3) + \frac{d}{dx}(\tan x)$

$[\because \frac{d}{dx}(\tan x) = \sec^2 x$ & $\frac{d}{dx}(x^n) = nx^{n-1}]$

$= 3x^2 + \sec^2 x$

$\therefore \frac{dy}{dx} = 3x^2 + \sec^2 x$

Differentiating again with respect to x :

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(3x^2 + \sec^2 x) = \frac{d}{dx}(3x^2) + \frac{d}{dx}(\sec^2 x)$$

$$\frac{d^2y}{dx^2} = 6x + 2\sec x \sec x \tan x$$

[differentiated $\sec^2 x$ using chain rule, let t = sec x and z = t^2 $\therefore \frac{dz}{dx} = \frac{dz}{dt} \times \frac{dt}{dx}$]

$$\frac{d^2y}{dx^2} = 6x + 2\sec^2 x \tan x$$

1 B. Question

Find the second order derivatives of each of the following functions:

sin (log x)

Answer

√Basic Idea: Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

√The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v, i.e. f = v(u(x)). For the sake of simplicity just assume t = u(x)

Then f = v(t). By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$$

√Product rule of differentiation- $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

√Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

Let's solve now:

Given, y = sin (log x)

We have to find $\frac{d^2y}{dx^2}$

As $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So lets first find dy/dx and differentiate it again.

$\therefore \frac{dy}{dx} = \frac{d}{dx}(\sin(\log x))$

differentiating $\sin(\log x)$ using the chain rule,

let, t = log x and y = sin t

$\because \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ [using chain rule]

$\frac{dy}{dx} = \cos t \times \frac{1}{x}$

$\frac{dy}{dx} = \cos(\log x) \times \frac{1}{x}$ [$\because \frac{d}{dx}$(log x) $= \frac{1}{x}$ & $\frac{d}{dx}(\sin x) = \cos x$]

Differentiating again with respect to x :

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\,(\cos(\log x) \times \frac{1}{x})$

$\frac{d^2y}{dx^2} = \cos(\log x) \times \frac{-1}{x^2} + \frac{1}{x} \times \frac{1}{x}\,(-\sin(\log x))$

[using product rule of differentiation]

$= \frac{-1}{x^2}\cos(\log x) - \frac{1}{x^2}\sin(\log x)$

$\frac{d^2y}{dx^2} = \frac{-1}{x^2}\cos(\log x) - \frac{1}{x^2}\sin(\log x)$

1 C. Question

Find the second order derivatives of each of the following functions:

log (sin x)

Answer

√Basic Idea: Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

√The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v, i.e. f = v(u(x)). For the sake of simplicity just assume t = u(x)

Then f = v(t). By chain rule, we can write the derivative of f w.r.t to x as:

$\frac{df}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$

√Product rule of differentiation- $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\,\frac{du}{dx}$

Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

Let's solve now:

Given, y = log (sin x)

We have to find $\frac{d^2y}{dx^2}$

As $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So lets first find dy/dx and differentiate it again.

$\therefore \frac{dy}{dx} = \frac{d}{dx}(\log(\sin x))$

differentiating $\sin(\log x)$ using cthe hain rule,

let, t = sin x and y = log t

$\because \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ [using chain rule]

$$\frac{dy}{dx} = \cos x \times \frac{1}{t}$$

[$\because \frac{d}{dx}\log x = \frac{1}{x}$ & $\frac{d}{dx}(\sin x) = \cos x$]

$$\frac{dy}{dx} = \frac{\cos x}{\sin x} = \cot x$$

Differentiating again with respect to x:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\cot x)$$

$\frac{d^2y}{dx^2} = -\operatorname{cosec}^2 x$ [$\because \frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$]

$$\frac{d^2y}{dx^2} = -\operatorname{cosec}^2 x$$

1 D. Question

Find the second order derivatives of each of the following functions:

$e^x \sin 5x$

Answer

√Basic Idea: Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

√The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v, i.e. f = v(u(x)). For the sake of simplicity just assume t = u(x)

Then f = v(t). By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$$

√Product rule of differentiation- $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

Let's solve now:

Given, $y = e^x \sin 5x$

We have to find $\frac{d^2y}{dx^2}$

As, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

So lets first find dy/dx and differentiate it again.

$\therefore \frac{dy}{dx} = \frac{d}{dx}(e^x \sin 5x)$

Let $u = e^x$ and $v = \sin 5x$

As, y = uv

∴ Using product rule of differentiation:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$\therefore \frac{dy}{dx} = e^x \frac{d}{dx}(\sin 5x) + \sin 5x \frac{d}{dx} e^x$

$$\frac{dy}{dx} = 5e^x \cos 5x + e^x \sin 5x$$

$[\because \frac{d}{dx}(\sin ax) = a\cos ax, \text{where a is any constant } \& \frac{d}{dx} e^x = e^x]$

Again differentiating w.r.t x:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(5e^x \cos 5x + e^x \sin 5x)$$

$$= \frac{d}{dx}(5e^x \cos 5x) + \frac{d}{dx}(e^x \sin 5x)$$

Again using the product rule :

$$\frac{d^2y}{dx^2} = e^x \frac{d}{dx}(\sin 5x) + \sin 5x \frac{d}{dx} e^x + 5e^x \frac{d}{dx}(\cos 5x) + \cos 5x \frac{d}{dx}(5e^x)$$

$\frac{d^2y}{dx^2} = 5e^x \cos 5x - 25e^x \sin 5x + e^x \sin 5x + 5e^x \cos 5x$ $[\because \frac{d}{dx}(\cos ax) = -a \sin ax, \text{a is any constant}]$

$$\frac{d^2y}{dx^2} = 10e^x \cos 5x - 24e^x \sin 5x$$

1 E. Question

Find the second order derivatives of each of the following functions:

e^{6x} cos 3x

Answer

√Basic Idea: Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

√The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v, i.e. f = v(u(x)). For the sake of simplicity just assume t = u(x)

Then f = v(t). By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$$

√Product rule of differentiation- $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

Let's solve now:

Given, y = e^{6x} cos 3x

We have to find $\frac{d^2y}{dx^2}$

As, $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So lets first find dy/dx and differentiate it again.

$\therefore \frac{dy}{dx} = \frac{d}{dx}(e^{6x} \cos 3x)$

Let u = e^{6x} and v = cos 3x

As, y = uv

∴ Using product rule of differentiation:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$\therefore \frac{dy}{dx} = e^{6x}\frac{d}{dx}(\cos 3x) + \cos 3x\frac{d}{dx}e^{6x}$

$\frac{dy}{dx} = -3e^{6x}\sin 3x + 6e^{6x}\cos 3x$ [$\because \frac{d}{dx}(\cos ax) = -a\sin ax$, a is any constant & $\frac{d}{dx}e^{ax} = ae^{x}$]

Again differentiating w.r.t x:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(-3e^{6x}\sin 3x + 6e^{6x}\cos 3x)$$

$$= \frac{d}{dx}(-3e^{6x}\sin 3x) + \frac{d}{dx}(6e^{6x}\cos 3x)$$

Again using the product rule :

$$\frac{d^2y}{dx^2} = -3e^{6x}\frac{d}{dx}(\sin 3x) - 3\sin 3x\frac{d}{dx}e^{6x} + 6e^{6x}\frac{d}{dx}(\cos 3x) + \cos 3x\frac{d}{dx}(6e^{6x})$$

$$\frac{d^2y}{dx^2} = -9e^{6x}\cos 3x - 18e^{6x}\sin 3x - 18e^{6x}\sin 3x + 36e^{6x}\cos 3x$$

$$\frac{d^2y}{dx^2} = 27e^{6x}\cos 3x - 36e^{6x}\sin 3x$$

1 F. Question

Find the second order derivatives of each of the following functions:

$x^3 \log x$

Answer

√Basic Idea: Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

√The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v, i.e. f = v(u(x)). For the sake of simplicity just assume t = u(x)

Then f = v(t). By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$$

√Product rule of differentiation- $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

Let's solve now:

Given, y = x^3 log x

We have to find $\frac{d^2y}{dx^2}$

As $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So lets first find dy/dx and differentiate it again.

$\therefore \frac{dy}{dx} = \frac{d}{dx}(x^3\log x)$

Let u = x^3 and v = log x

As, y = uv

∴ Using product rule of differentiation:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\therefore \frac{dy}{dx} = x^3\frac{d}{dx}(\log x) + \log x\frac{d}{dx}x^3$$

$$\frac{dy}{dx} = 3x^2\log x + \frac{x^3}{x}$$

$[\because \frac{d}{dx}(\log x) = \frac{1}{x} \text{ and } \frac{d}{dx}(x^n) = nx^{n-1}]$

Again differentiating w.r.t x:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(3x^2\log x + x^2)$$

$$= \frac{d}{dx}(3x^2\log x) + \frac{d}{dx}(x^2)$$

Again using the product rule :

$$\frac{d^2y}{dx^2} = 3\log x\frac{d}{dx}x^2 + 3x^2\frac{d}{dx}\log x + \frac{d}{dx}x^2$$

$[\because \frac{d}{dx}(\log x) = \frac{1}{x} \text{ and } \frac{d}{dx}(x^n) = nx^{n-1}]$

$$\frac{d^2y}{dx^2} = 6x\log x + \frac{3x^2}{x} + 2x$$

$$\frac{d^2y}{dx^2} = 6x\log x + 5x$$

1 G. Question

Find the second order derivatives of each of the following functions:

$\tan^{-1} x$

Answer

<u>Basic idea:</u>

√Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

√The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v, i.e. f = v(u(x)). For the sake of simplicity just assume t = u(x)

Then f = v(t). By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx} = \frac{dv}{dt}\times\frac{dt}{dx}$$

√Product rule of differentiation- $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

√Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

Let's solve now:

Given, y = $\tan^{-1} x$

We have to find $\frac{d^2y}{dx^2}$

As $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So lets first find dy/dx and differentiate it again.

$\therefore \frac{dy}{dx} = \frac{d}{dx}(\tan^{-1}x)$ [$\because \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$]

$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}$ [$\because \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$]

Differentiating again with respect to x :

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{1+x^2}\right)$$

Differentiating $\frac{1}{1+x^2}$ using chain rule,

let t = 1 +x^2 and z = 1/t

$\because \frac{dz}{dx} = \frac{dz}{dt} \times \frac{dt}{dx}$ [from chain rule of differentiation]

$\therefore \frac{dz}{dx} = \frac{-1}{t^2} \times 2x = -\frac{2x}{1+x^2}$ [$\because \frac{d}{dx}(x^n) = nx^{n-1}$]

$\therefore \frac{d^2y}{dx^2} = -\frac{2x}{(1+x^2)^2}$

1 H. Question

Find the second order derivatives of each of the following functions:

x cos x

Answer

√Basic Idea: Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

√The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v, i.e. f = v(u(x)). For the sake of simplicity just assume t = u(x)

Then f = v(t). By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$$

√Product rule of differentiation- $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

Let's solve now:

Given, y = x cos x

We have to find $\frac{d^2y}{dx^2}$

As $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So lets first find dy/dx and differentiate it again.

$\therefore \frac{dy}{dx} = \frac{d}{dx}(x\cos x)$

Let u = x and v = cos x

As, y = uv

∴ Using product rule of differentiation:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\therefore \frac{dy}{dx} = x\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}x$$

$$\frac{dy}{dx} = -x\sin x + \cos x$$

$[\because \frac{d}{dx}(\cos x) = -\sin x \text{ and } \frac{d}{dx}(x^n) = nx^{n-1}]$

Again differentiating w.r.t x:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(-x\sin x + \cos x)$$

$$= \frac{d}{dx}(-x\sin x) + \frac{d}{dx}\cos x$$

Again using the product rule :

$$\frac{d^2y}{dx^2} = -x\frac{d}{dx}\sin x + \sin x\frac{d}{dx}(-x) + \frac{d}{dx}\cos x$$

$[\because \frac{d}{dx}(\sin x) = \cos x \text{ and } \frac{d}{dx}(x^n) = nx^{n-1}]$

$$\frac{d^2y}{dx^2} = -x\cos x - \sin x - \sin x$$

$$\frac{d^2y}{dx^2} = -x\cos x - 2\sin x$$

1 I. Question

Find the second order derivatives of each of the following functions:

log (log x)

Answer

√Basic Idea: Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

√The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v, i.e. f = v(u(x)). For the sake of simplicity just assume t = u(x)

Then f = v(t). By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$$

√Product rule of differentiation- $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

Let’s solve now:

Given, y = log (log x)

We have to find $\frac{d^2y}{dx^2}$

As, $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So lets first find dy/dx and differentiate it again.

$\therefore \frac{dy}{dx} = \frac{d}{dx}(\log\log x)$

Let y = log t and t = log x

Using chain rule of differentiation:

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$\therefore \frac{dy}{dx} = \frac{1}{t} \times \frac{1}{x} = \frac{1}{x\log x}$ [$\because \frac{d}{dx}$(log x) $= \frac{1}{x}$]

Again differentiating w.r.t x:

As, $\frac{dy}{dx} = u \times v$

Where u $= \frac{1}{x}$ and v $= \frac{1}{\log x}$

∴ using product rule of differentiation:

$$\frac{d^2y}{dx^2} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$\therefore \frac{d^2y}{dx^2} = \frac{1}{x}\frac{d}{dx}\left(\frac{1}{\log x}\right) + \frac{1}{\log x}\frac{d}{dx}\left(\frac{1}{x}\right)$ [use chain rule to find $\frac{d}{dx}\left(\frac{1}{\log x}\right)$]

$\frac{d^2y}{dx^2} = -\frac{1}{x^2(\log x)^2} - \frac{1}{x^2\log x}$ [$\because \frac{d}{dx}(\log x) = \frac{1}{x}$ and $\frac{d}{dx}(x^n) = nx^{n-1}$]

$$\therefore \frac{d^2y}{dx^2} = -\frac{1}{x^2(\log x)^2} - \frac{1}{x^2\log x}$$

2. Question

If y=e^{-x} cos x, show that : $\frac{d^2y}{dx^2} = 2e^{-x}$ sin x.

Answer

Basic idea:

√Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

√The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v, i.e. f = v(u(x)). For the sake of simplicity just assume t = u(x)

Then f = v(t). By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$$

√Product rule of differentiation- $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

√Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

Let's solve now:

Given,

y=e^{-x} cos x

TO prove :

$$\frac{d^2y}{dx^2} = 2e^{-x} \sin x.$$

Clearly from the expression to be proved we can easily observe that we need to just find the second derivative of given function.

Given, $y = e^{-x} \cos x$

We have to find $\frac{d^2y}{dx^2}$

As, $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So lets first find dy/dx and differentiate it again.

$\therefore \frac{dy}{dx} = \frac{d}{dx}(e^{-x}\cos x)$

Let $u = e^{-x}$ and $v = \cos x$

As, $y = u*v$

$\therefore$ using product rule of differentiation:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$\therefore \frac{dy}{dx} = e^{-x}\frac{d}{dx}(\cos x) + \cos x\frac{dy}{dx}e^{-x}$

$$\frac{dy}{dx} = -e^{-x}\sin x - e^{-x}\cos x$$

$[\because \frac{d}{dx}(\cos x) = -\sin x \ \& \ \frac{d}{dx}e^{-x} = -e^{-x}]$

Again differentiating w.r.t x:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(-e^{-x}\sin x - e^{-x}\cos x)$$

$$= \frac{d}{dx}(-e^{-x}\sin x) - \frac{d}{dx}(e^{-x}\cos x)$$

Again using the product rule :

$$\frac{d^2y}{dx^2} = -e^{-x}\frac{d}{dx}(\sin x) - \sin x\frac{d}{dx}e^{-x} - e^{-x}\frac{d}{dx}(\cos x) - \cos x\frac{d}{dx}(e^{-x})$$

$$\frac{d^2y}{dx^2} = -e^{-x}\cos x + e^{-x}\sin x + e^{-x}\sin x + e^{-x}\cos x$$

$[\because \frac{d}{dx}(\cos x) = -\sin x, \frac{d}{dx}e^{-x} = -e^{-x}]$

$$\frac{d^2y}{dx^2} = 2e^{-x}\sin x \ \ldots\text{proved}$$

3. Question

If y = x + tan x, show that: $\cos^2 x\frac{d^2y}{dx^2} - 2y - 2x = 0$

Answer

Basic idea:

√Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

√The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v, i.e. f = v(u(x)). For the sake of simplicity just assume t = u(x)

Then f = v(t). By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$$

√Product rule of differentiation- $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

√Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

Let's solve now:

Given, y = x+ tan x .....equation 1

As we have to prove: $\cos^2 x \frac{d^2y}{dx^2} - 2y + 2x = 0$

We notice a second-order derivative in the expression to be proved so first take the step to find the second order derivative.

Let's find $\frac{d^2y}{dx^2}$

As $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So lets first find dy/dx and differentiate it again.

$\therefore \frac{dy}{dx} = \frac{d}{dx}(x + \tan x) = \frac{d}{dx}(x) + \frac{d}{dx}(\tan x)$ [$\because \frac{d}{dx}(\tan x) = \sec^2 x$ & $\frac{d}{dx}(x^n) = nx^{n-1}$]

$= 1 + \sec^2 x$

$\therefore \frac{dy}{dx} = 1 + \sec^2 x$

Differentiating again with respect to x :

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(1 + \sec^2 x) = \frac{d}{dx}(1) + \frac{d}{dx}(\sec^2 x)$$

$$\frac{d^2y}{dx^2} = 0 + 2\sec x \sec x \tan x$$

[differentiated $\sec^2 x$ using chain rule, let t = sec x and z = t^2 $\therefore \frac{dz}{dx} = \frac{dz}{dt} \times \frac{dt}{dx}$]

$\frac{d^2y}{dx^2} = 2\sec^2 x \tan x$ ....equation 2

As we got an expression for the second order, as we need $\cos^2 x$ term with $\frac{d^2y}{dx^2}$

Multiply both sides of equation 1 with $\cos^2 x$:

∴ we have,

$\cos^2 x \frac{d^2y}{dx^2} = 2\cos^2 x \sec^2 x \tan x$ [∵ cos x × sec x = 1]

$$\cos^2 x \frac{d^2y}{dx^2} = 2\tan x$$

From equation 1:

tan x = y - x∴ $\cos^2 x \frac{d^2y}{dx^2} = 2(y - x)$

∴ $\cos^2 x \frac{d^2y}{dx^2} - 2y + 2x = 0$ ....proved

4. Question

If y = x^3 log x, prove that $\frac{d^4y}{dx^4} = \frac{6}{x}$.

Answer

Basic idea:

√Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

√The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v, i.e. f = v(u(x)). For the sake of simplicity just assume t = u(x)

Then f = v(t). By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$$

√Product rule of differentiation- $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\ \frac{du}{dx}$

√Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

Let's solve now:

As we have to prove : $\frac{d^4y}{dx^4} = \frac{6}{x}$

We notice a third order derivative in the expression to be proved so first take the step to find the third order derivative.

Given, y = x^3 log x

Let's find - $\frac{d^4y}{dx^4}$

As $\frac{d^4y}{dx^4} = \frac{d}{dx}(\frac{d^3y}{dx^3}) = \frac{d}{dx}\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right)\right)$

So lets first find dy/dx and differentiate it again.

∴ $\frac{dy}{dx} = \frac{d}{dx}(x^3 \log x)$

differentiating using product rule:

$$\frac{dy}{dx} = x^3\frac{d}{dx}\log x + \log x\frac{d}{dx}x^3$$

$$\frac{dy}{dx} = \frac{x^3}{x} + 3x^2 \log x$$

[$\frac{d}{dx}(x^n) = nx^{n-1}$ & $\frac{d}{dx}$(log x) = $\frac{1}{x}$]

$$\frac{dy}{dx} = x^2(1 + 3\log x)$$

Again differentiating using product rule:

$$\frac{d^2y}{dx^2} = x^2\frac{d}{dx}(1+3\log x) + (1+3\log x)\frac{d}{dx}x^2$$

$$\frac{d^2y}{dx^2} = x^2 \times \frac{3}{x} + (1+3\log x) \times 2x$$

$[\frac{d}{dx}(x^n) = nx^{n-1}$ & $\frac{d}{dx}(\log x) = \frac{1}{x}]$

$$\frac{d^2y}{dx^2} = x(5+6\log x)$$

Again differentiating using product rule:

$$\frac{d^3y}{dx^3} = x\frac{d}{dx}(5+6\log x) + (5+6\log x)\frac{d}{dx}x$$

$$\frac{d^3y}{dx^3} = x \times \frac{6}{x} + (5+6\log x)$$

$[\frac{d}{dx}(x^n) = nx^{n-1}$ & $\frac{d}{dx}(\log x) = \frac{1}{x}]$

$$\frac{d^3y}{dx^3} = 11 + 6\log x$$

Again differentiating w.r.t x :

$$\frac{d^4y}{dx^4} = \frac{6}{x} \text{ ……. proved}$$

5. Question

If y = log (sin x), prove that: $\frac{d^3y}{dx^2} = 2\cos x \operatorname{cose}^3 x$.

Answer

Basic idea:

√Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

√The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v, i.e. f = v(u(x)). For the sake of simplicity just assume t = u(x)

Then f = v(t). By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$$

√Product rule of differentiation- $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

√Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

Let’s solve now:

As we have to prove: $\frac{d^3y}{dx^2} = 2\cos x \operatorname{cose}^3 x$

We notice a third order derivative in the expression to be proved so first take the step to find the third order derivative.

Given, y = log (sin x)

Let's find – $\frac{d^3y}{dx^3}$

As $\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right)$

So lets first find dy/dx and differentiate it again.

$\therefore \frac{dy}{dx} = \frac{d}{dx}(\log(\sin x))$

differentiating $\sin(\log x)$ using the chain rule,

let, t = sin x and y = log t

$\because \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ [using chain rule]

$$\frac{dy}{dx} = \cos x \times \frac{1}{t}$$

$[\because \frac{d}{dx}\log x = \frac{1}{x}$ & $\frac{d}{dx}(\sin x) = \cos x]$

$$\frac{dy}{dx} = \frac{\cos x}{\sin x} = \cot x$$

Differentiating again with respect to x :

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\cot x)$$

$$\frac{d^2y}{dx^2} = -\operatorname{cosec}^2 x$$

$[\because \frac{d}{dx}\cot x = -\operatorname{cosec}^2 x]$

$$\frac{d^2y}{dx^2} = -\operatorname{cosec}^2 x$$

Differentiating again with respect to x:

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}(-\operatorname{cosec}^2 x)$$

using the chain rule and $\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot x$

$$\frac{d^3y}{dx^3} = -2\operatorname{cosec} x(-\operatorname{cosec} x \cot x)$$

$= 2\operatorname{cosec}^2 x \cot x = 2\operatorname{cosec}^2 x \frac{\cos x}{\sin x}$ [∵ cot x = cos x/sin x]

$$\therefore \frac{d^3y}{dx^3} = 2\operatorname{cosec}^3 x \cos x \text{ ......proved}$$

6. Question

If y = 2 sin x + 3 cos x, show that: $\frac{d^2y}{dx^2} + y = 0$

Answer

<u>Basic idea:</u>

√Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

√The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v, i.e. f = v(u(x)). For the sake of simplicity just assume t = u(x)

Then f = v(t). By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx}=\frac{dv}{dt}\times\frac{dt}{dx}$$

√Product rule of differentiation- $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$

√Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

Let’s solve now:

Given, y = 2sin x+ 3cos x .....equation 1

As we have to prove : $\frac{d^2y}{dx^2}+y=0$.

We notice a second-order derivative in the expression to be proved so first take the step to find the second order derivative.

Let’s find $\frac{d^2y}{dx^2}$

As $\frac{d^2y}{dx^2}=\frac{d}{dx}(\frac{dy}{dx})$

So lets first find dy/dx and differentiate it again.

$\therefore \frac{dy}{dx}=\frac{d}{dx}(2\sin x+3\cos x)=2\frac{d}{dx}(\sin x)+3\frac{d}{dx}(\cos x)$

[$\because \frac{d}{dx}$(sin x) = cosx & $\frac{d}{dx}(\cos x)=-\sin x$]

$=2\cos x-3\sin x$

$\therefore \frac{dy}{dx}=2\cos x-3\sin x$

Differentiating again with respect to x :

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}(2\cos x-3\sin x)=\frac{2d}{dx}\cos x-3\frac{d}{dx}\sin x$$

$$\frac{d^2y}{dx^2}=-2\sin x-3\cos x$$

From equation 1 we have :

y = 2 sin x + 3 cos x

$\therefore \frac{d^2y}{dx^2}=-(2\sin x+3\cos x)=-y$

$\therefore \frac{d^2y}{dx^2}+y=0$ .....proved

7. Question

If y = $\frac{\log x}{x}$, show that $\frac{d^2y}{dx^2}=\frac{2\log x-3}{x^3}$.

Answer

Basic idea:

√Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2y}{dx^2}=\frac{d}{dx}(\frac{dy}{dx})$

√The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v, i.e. f = v(u(x)). For the sake of simplicity just assume t = u(x)

Then f = v(t). By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$$

√Product rule of differentiation- $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\ \frac{du}{dx}$

√Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

Let’s solve now:

Given, $y = \frac{\log x}{x}$ .....equation 1

As we have to prove : $\frac{d^2y}{dx^2} = \frac{2\log x - 3}{x^3}$..

We notice a second-order derivative in the expression to be proved so first take the step to find the second order derivative.

Let’s find $\frac{d^2y}{dx^2}$

As $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So, lets first find dy/dx and differentiate it again.

As y is the product of two functions u and v

Let u = log x and v = 1/x

Using product rule of differentiation:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\ \frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{\log x}{x}\right) = \log x \frac{d}{dx}\frac{1}{x} + \frac{1}{x}\frac{d}{dx}\log x$$

[∵ $\frac{d}{dx}(\log x) = \frac{1}{x}$ & $\frac{d}{dx}(x^n) = nx^{n-1}$]

$$\frac{dy}{dx} = -\frac{1}{x^2}\log x + \frac{1}{x^2}$$

$$\frac{dy}{dx} = \frac{1}{x^2}\ (1 - \log x)$$

Again using the product rule to find $\frac{d^2y}{dx^2}$:

$$\frac{d^2y}{dx^2} = (1 - \log x)\frac{d}{dx}\frac{1}{x^2} + \frac{1}{x^2}\frac{d}{dx}(1 - \log x)$$

[∵ $\frac{d}{dx}(\log x) = \frac{1}{x}$ & $\frac{d}{dx}(x^n) = nx^{n-1}$]

$$= -2\left(\frac{1 - \log x}{x^3}\right) - \frac{1}{x^3}$$

∴ $\frac{d^2y}{dx^2} = \frac{2\log x - 3}{x^3}$ ..... proved

8. Question

If x = a sec θ, y = b tan θ, prove that $\frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}$.

Answer

Idea of parametric form of differentiation:

If y = f (θ) and x = g(θ) i.e. y is a function of θ and x is also some other function of θ.

Then dy/dθ = f'(θ) and dx/dθ = g'(θ)

We can write : $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

Given,

x = a sec θ ......equation 1

y = b tan θ ......equation 2

to prove : $\frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}$.

We notice a second order derivative in the expression to be proved so first take the step to find the second order derivative.

Let's find $\frac{d^2y}{dx^2}$

As, $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So, lets first find dy/dx using parametric form and differentiate it again.

$\frac{dx}{d\theta} = \frac{d}{d\theta} a \sec\theta = a \sec\theta \tan\theta$ .....equation 3

Similarly, $\frac{dy}{d\theta} = b \sec^2\theta$ ......equation 4

$[\because \frac{d}{dx}\sec x = \sec x \tan x, \frac{d}{dx}\tan x = \sec^2 x]$

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b}{a}\operatorname{cosec}\theta$

Differentiating again w.r.t x :

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\frac{b}{a}\operatorname{cosec}\theta)$

$\frac{d^2y}{dx^2} = -\frac{b}{a}\operatorname{cosec}\theta \cot\theta \frac{d\theta}{dx}$ .....equation 5 [using chain rule]

From equation 3:

$\frac{dx}{d\theta} = a \sec\theta \tan\theta$

$\therefore \frac{d\theta}{dx} = \frac{1}{a \sec\theta \tan\theta}$

Putting the value in equation 5 :

$\frac{d^2y}{dx^2} = -\frac{b}{a}\operatorname{cosec}\theta \cot\theta \frac{1}{a\sec\theta\tan\theta}$

$\frac{d^2y}{dx^2} = \frac{-b}{a^2\tan^3\theta}$

From equation 1:

y = b tan θ

$\therefore \frac{d^2y}{dx^2} = \frac{-b}{\frac{a^2y^3}{b^3}} = -\frac{b^4}{a^2y^3}$ .....proved.

9. Question

If x =a (cos θ + θ sin θ), y=a (sin θ – θ cos θ) prove that

$\frac{d^2x}{d\theta^2} = a(\cos\theta - \theta\sin\theta), \frac{d^2y}{d\theta^2} = a$ (sin θ + θ cosθ) and $\frac{d^2y}{dx^2} = \frac{\sec^3\theta}{a\theta}$.

Answer

Basic idea:

√Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

√The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v, i.e. f = v(u(x)). For the sake of simplicity just assume t = u(x)

Then f = v(t). By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$$

√Product rule of differentiation- $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

√Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

The idea of parametric form of differentiation:

If y = f (θ) and x = g(θ), i.e. y is a function of θ and x is also some other function of θ.

Then dy/dθ = f’(θ) and dx/dθ = g’(θ)

We can write : $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

Given,

x =a (cos θ + θ sin θ) ......equation 1

y =a (sin θ – θ cos θ) ......equation 2

to prove :

i) $\frac{d^2x}{d\theta^2} = a(\cos\theta - \theta\sin\theta)$

ii) $\frac{d^2y}{d\theta^2} = a$ (sin θ + θ cosθ)

iii) $\frac{d^2y}{dx^2} = \frac{\sec^3\theta}{a\theta}$.

We notice a second order derivative in the expression to be proved so first take the step to find the second order derivative.

Let’s find $\frac{d^2y}{dx^2}$

As $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

$$\frac{dx}{d\theta} = \frac{d}{d\theta}a(\cos\theta + \theta\sin\theta)$$

$= a(-\sin\theta + \theta\cos\theta + \sin\theta)$

[differentiated using product rule for θsinθ]

$= a\theta\cos\theta$ ..eqn 4

Again differentiating w.r.t θ using product rule:-

$$\frac{d^2x}{d\theta^2} = a(-\theta\sin\theta + \cos\theta)$$

$\therefore \frac{d^2x}{d\theta^2} = a(\cos\theta - \theta\sin\theta)$ .... proved (i)

Similarly,

$$\frac{dy}{d\theta} = \frac{d}{d\theta}a(\sin\theta - \theta\cos\theta) = a\frac{d}{d\theta}\sin\theta - a\frac{d}{d\theta}(\theta\cos\theta)$$

$= a\cos\theta + a\theta\sin\theta - a\cos\theta$

$\therefore \frac{dy}{d\theta} = a\theta\sin\theta$ ....equation 5

Again differentiating w.r.t θ using product rule:-

$$\frac{d^2x}{d\theta^2} = a(\theta\cos\theta + \sin\theta)$$

$\therefore \frac{d^2x}{d\theta^2} = a(\sin\theta + \theta\cos\theta)$ .... proved (ii)

$\because \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

Using equation 4 and 5 :

$$\frac{dy}{dx} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

As $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

∴ again differentiating w.r.t x :-

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\tan\theta$$

$= \sec^2\theta\frac{d\theta}{dx}$ [using chain rule]

$\because \frac{dx}{d\theta} = a\theta\cos\theta \Rightarrow \frac{d\theta}{dx} = \frac{1}{a\theta\cos\theta}$

Putting a value in the above equation-

We have :

$$\frac{d^2y}{dx^2} = \sec^2\theta \times \frac{1}{a\theta\cos\theta}$$

$$\frac{d^2y}{dx^2} = \frac{\sec^3\theta}{a\theta} \text{ ...... proved (iii)}$$

10. Question

If y = e^x cosx, prove that $\frac{d^2y}{dx^2} = 2e^x\cos\left(x + \frac{\pi}{2}\right)$

Answer

Basic idea:

√Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

√The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v, i.e. f = v(u(x)). For the sake of simplicity just assume t = u(x)

Then f = v(t). By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$$

√Product rule of differentiation- $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

√Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

Let's solve now:

Given,

$y=e^x \cos x$

TO prove :

$$\frac{d^2y}{dx^2} = 2e^x \cos\left(x + \frac{\pi}{2}\right)$$

Clearly from the expression to be proved we can easily observe that we need to just find the second derivative of given function.

Given, $y = e^x \cos x$

We have to find $\frac{d^2y}{dx^2}$

As $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So lets first find dy/dx and differentiate it again.

$\therefore \frac{dy}{dx} = \frac{d}{dx}(e^x \cos x)$

Let $u = e^x$ and $v = \cos x$

As, y = u*v

∴ Using product rule of differentiation:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$\therefore \frac{dy}{dx} = e^x \frac{d}{dx}(\cos x) + \cos x \frac{dy}{dx} e^x$

$\frac{dy}{dx} = -e^x \sin x + e^x \cos x$ [$\because \frac{d}{dx}(\cos x) = -\sin x$ & $\frac{d}{dx}e^x = e^x$]

Again differentiating w.r.t x:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(-e^x \sin x + e^x \cos x)$$

$$= \frac{d}{dx}(-e^x \sin x) + \frac{d}{dx}(e^x \cos x)$$

Again using the product rule :

$$\frac{d^2y}{dx^2} = -e^x\frac{d}{dx}(\sin x) - \sin x\frac{d}{dx}e^x + e^x\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}(e^x)$$

$$\frac{d^2y}{dx^2} = -e^x\cos x - e^x\sin x - e^x\sin x + e^x\cos x$$

$[\because \frac{d}{dx}(\cos x) = -\sin x, \frac{d}{dx}e^{-x} = -e^{-x}]$

$\frac{d^2y}{dx^2} = -2e^x\sin x$ [∵ –sin x = cos (x + π/2)]

$$\frac{d^2y}{dx^2} = -2e^x\cos(x+\frac{\pi}{2}) \text{ ....proved}$$

11. Question

If x = a cos θ , y = b sin θ, show that $\frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}$.

Answer

Idea of parametric form of differentiation:

If y = f (θ) and x = g(θ) i.e. y is a function of θ and x is also some other function of θ.

Then dy/dθ = f'(θ) and dx/dθ = g'(θ)

We can write : $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

Given,

x = a cos θ ......equation 1

y = b sin θ ......equation 2

to prove : $\frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}$.

We notice a second order derivative in the expression to be proved so first take the step to find the second order derivative.

Let's find $\frac{d^2y}{dx^2}$

As $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So, lets first find dy/dx using parametric form and differentiate it again.

$\frac{dx}{d\theta} = \frac{d}{d\theta}a\cos\theta = -a\sin\theta$ .....equation 3

Similarly, $\frac{dy}{d\theta} = b\cos\theta$ ......equation 4

$[\because \frac{d}{dx}\cos x = -\sin x \tan x, \frac{d}{dx}\sin x = \cos x$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\frac{b\cos\theta}{a\sin\theta} = -\frac{b}{a}\cot\theta$$

Differentiating again w.r.t x :

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(-\frac{b}{a}\cot\theta)$$

$\frac{d^2y}{dx^2} = \frac{b}{a}\text{cosec}^2\theta\frac{d\theta}{dx}$ .....equation 5

[using chain rule and $\frac{d}{dx}\cot x = -\text{cosec}^2 x$]

From equation 3:

$$\frac{dx}{d\theta} = -a\sin\theta$$

$\therefore \frac{d\theta}{dx} = \frac{-1}{a\sin\theta}$

Putting the value in equation 5 :

$$\frac{d^2y}{dx^2} = -\frac{b}{a}\text{cosec}^2\theta\frac{1}{a\sin\theta}$$

$$\frac{d^2y}{dx^2} = \frac{-b}{a^2\sin^3\theta}$$

From equation 1:

y = b sin θ

$\therefore \frac{d^2y}{dx^2} = \frac{-b}{\frac{a^2y^3}{b^3}} = -\frac{b^4}{a^2y^3}$ .....proved.

12. Question

If x = a (1 – cos ^{3}θ), y = a sin ^{3} θ, Prove that $\frac{d^2y}{dx^2} = \frac{32}{27\,a}$ at $\theta = \frac{\pi}{6}$.

Answer

Idea of parametric form of differentiation:

If y = f (θ) and x = g(θ) i.e. y is a function of θ and x is also some other function of θ.

Then dy/dθ = f'(θ) and dx/dθ = g'(θ)

We can write : $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

Given,

x = a (1 – cos ^{3}θ) ......equation 1

y = a sin ^{3} θ, ...equation 2

to prove : $\frac{d^2y}{dx^2} = \frac{32}{27a}$ at $\theta = \frac{\pi}{6}$.

We notice a second order derivative in the expression to be proved so first take the step to find the second order derivative.

Let's find $\frac{d^2y}{dx^2}$

As $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

So, lets first find dy/dx using parametric form and differentiate it again.

$\frac{dx}{d\theta} = \frac{d}{d\theta}a\,(1 - \cos^3\theta) = 3a\cos^2\theta\sin\theta$ .....equation 3 [using chain rule]

Similarly,

$\frac{dy}{d\theta} = \frac{d}{d\theta} a\sin^3\theta = 3a\sin^2\theta\cos\theta$ ......equation 4

$[\because \frac{d}{dx}\cos x = -\sin x \;\&\; \frac{d}{dx}\cos x = \sin x]$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a\sin^2\theta\cos\theta}{3a\cos^2\theta\sin\theta} = \tan\theta$$

Differentiating again w.r.t x :

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\tan\theta)$$

$\frac{d^2y}{dx^2} = \sec^2\theta\frac{d\theta}{dx}$ .....equation 5

[using chain rule and $\frac{d}{dx}\tan x = \sec^2 x$]

From equation 3:

$$\frac{dx}{d\theta} = 3a\cos^2\theta\sin\theta$$

$$\therefore \frac{d\theta}{dx} = \frac{1}{3a\cos^2\theta\sin\theta}$$

Putting the value in equation 5 :

$$\frac{d^2y}{dx^2} = \sec^2\theta\frac{1}{3a\cos^2\theta\sin\theta}$$

$$\frac{d^2y}{dx^2} = \frac{1}{3a\cos^4\theta\sin\theta}$$

Put θ = π/6

$$\left(\frac{d^2y}{dx^2}\right) at \left(x = \frac{\pi}{6}\right) = \frac{1}{3a\cos^4\frac{\pi}{6}\sin\frac{\pi}{6}} = \frac{1}{3a\left(\frac{\sqrt{3}}{2}\right)^4\frac{1}{2}}$$

$$\therefore \left(\frac{d^2y}{dx^2}\right) at \left(x = \frac{\pi}{6}\right) = \frac{32}{27a} \text{ .... proved}$$

13. Question

If x = a (θ + sin θ), y = a (1+ cos θ), prove that $\frac{d^2y}{dx^2} = -\frac{a}{y^2}$.

Answer

Idea of parametric form of differentiation:

If y = f (θ) and x = g(θ) i.e. y is a function of θ and x is also some other function of θ.

Then dy/dθ = f'(θ) and dx/dθ = g'(θ)

We can write : $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

Given,

x = a (θ + sin θ) ......equation 1

y = a (1+ cos θ) ......equation 2

to prove : $\frac{d^2y}{dx^2} = -\frac{a}{y^2}$

We notice a second-order derivative in the expression to be proved so first take the step to find the second order derivative.

Let's find $\frac{d^2y}{dx^2}$

As, $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So, lets first find dy/dx using parametric form and differentiate it again.

$\frac{dx}{d\theta} = \frac{d}{d\theta}a\,(\theta + \sin\theta) = a(1+\cos\theta) = y$ [∵ from equation 2] .....equation 3

Similarly,

$\frac{dy}{d\theta} = \frac{d}{d\theta}a\,(1+ \cos\theta) = -a\sin\theta$ ......equation 4

[∵ $\frac{d}{dx}\cos x = -\sin x, \frac{d}{dx}\sin x = \cos x$

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-a\sin\theta}{a(1+\cos\theta)} = \frac{-\sin\theta}{(1+\cos\theta)} = \frac{-a\sin\theta}{y}$ [∵ from equation 2] .....equation 5

Differentiating again w.r.t x :

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = -a\frac{d}{dx}\left(\frac{\sin\theta}{y}\right)$$

Using product rule and chain rule of differentiation together:

$$\frac{d^2y}{dx^2} = -a\left(\frac{\sin\theta}{-y^2}\frac{dy}{dx} + \frac{1}{y}\cos\theta\frac{d\theta}{dx}\right)$$

$\frac{d^2y}{dx^2} = -a\left(\frac{\sin\theta}{-y^2}\frac{(-a\sin\theta)}{y} + \frac{1}{y}\cos\theta\frac{1}{y}\right)$ [using equation 3 and 5]

$$\frac{d^2y}{dx^2} = -a\left(\frac{a\sin^2\theta}{y^3} + \frac{1}{y^2}\cos\theta\right)$$

$\frac{d^2y}{dx^2} = -\frac{a}{y^2}\left(\frac{a\sin^2\theta}{a\,(1+\cos\theta)} + \cos\theta\right)$ [from equation 1]

$$\frac{d^2y}{dx^2} = -\frac{a}{y^2}\left(\frac{1-\cos^2\theta}{(1+\cos\theta)} + \cos\theta\right)$$

$$\frac{d^2y}{dx^2} = -\frac{a}{y^2}\left(\frac{(1-\cos\theta)(1+\cos\theta)}{(1+\cos\theta)} + \cos\theta\right)$$

$$\frac{d^2y}{dx^2} = -\frac{a}{y^2}(1-\cos\theta + \cos\theta)$$

$\therefore \frac{d^2y}{dx^2} = -\frac{a}{y^2}$ ....proved

14. Question

If x = a (θ – sin θ), y = a (1 + cos θ) find $\frac{d^2y}{dx^2}$.

Answer

Idea of parametric form of differentiation:

If y = f (θ) and x = g(θ) i.e. y is a function of θ and x is also some other function of θ.

Then dy/dθ = f'(θ) and dx/dθ = g'(θ)

We can write : $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

Given,

x = a (θ – sin θ) ......equation 1

y = a (1+ cos θ) ......equation 2

to find : $\frac{d^2y}{dx^2}$

As, $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So, lets first find dy/dx using parametric form and differentiate it again.

$\frac{dx}{d\theta} = \frac{d}{d\theta}a\,(\theta - \sin\theta) = a(1 - \cos\theta)$ .....equation 3

Similarly,

$\frac{dy}{d\theta} = \frac{d}{d\theta}a\,(1 + \cos\theta) = -a\sin\theta$ ......equation 4

$[\because \frac{d}{dx}\cos x = -\sin x, \frac{d}{dx}\sin x = \cos x$

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-a\sin\theta}{a(1-\cos\theta)} = \frac{-\sin\theta}{(1-\cos\theta)}$ .....equation 5

Differentiating again w.r.t x :

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = -\frac{d}{dx}(\frac{\sin\theta}{1 - \cos\theta})$$

Using product rule and chain rule of differentiation together:

$$\frac{d^2y}{dx^2} = \{-\frac{1}{1 - \cos\theta}\frac{d}{d\theta}\sin\theta - \sin\theta\frac{d}{d\theta}\frac{1}{(1 - \cos\theta)}\}\frac{d\theta}{dx}$$

Apply chain rule to determine $\frac{d}{d\theta}\frac{1}{(1-\cos\theta)}$

$\frac{d^2y}{dx^2} = \left\{\frac{-\cos\theta}{1-\cos\theta} + \frac{\sin^2\theta}{(1-\cos\theta)^2}\right\}\frac{1}{a(1-\cos\theta)}$ [using equation 3]

$$\frac{d^2y}{dx^2} = \left\{\frac{-\cos\theta(1 - \cos\theta) + \sin^2\theta}{(1 - \cos\theta)^2}\right\}\frac{1}{a(1 - \cos\theta)}$$

$$\frac{d^2y}{dx^2} = \left\{\frac{-\cos\theta + \cos^2\theta + \sin^2\theta}{(1 - \cos\theta)^2}\right\}\frac{1}{a(1 - \cos\theta)}$$

$\frac{d^2y}{dx^2} = \left\{\frac{1-\cos\theta}{(1-\cos\theta)^2}\right\}\frac{1}{a(1-\cos\theta)}$ $[\because \cos^2\theta + \sin^2\theta = 1]$

$$\frac{d^2y}{dx^2} = \frac{1}{a(1 - \cos\theta)^2}$$

$\frac{d^2y}{dx^2} = \frac{1}{a\left(2\sin^2\frac{\theta}{2}\right)^2}$ [∵1- cos θ = 2sin^2 θ/2]

$\therefore \frac{d^2y}{dx^2} = \frac{1}{4a}\operatorname{cosec}^4\frac{\theta}{2}$

15. Question

If x = a (1 – cos θ), y =a (θ + sin θ), prove that $\frac{d^2y}{dx^2} = -\frac{1}{a}$at $\theta = \frac{\pi}{2}$

Answer

Idea of parametric form of differentiation:

If y = f (θ) and x = g(θ) i.e. y is a function of θ and x is also some other function of θ.

Then dy/dθ = f'(θ) and dx/dθ = g'(θ)

We can write : $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

Given,

y = a (θ + sin θ) ……equation 1

x = a (1– cos θ) ……equation 2

to prove : $\frac{d^2y}{dx^2} = -\frac{1}{a}$ at $\theta = \frac{\pi}{2}$.

We notice a second-order derivative in the expression to be proved so first take the step to find the second order derivative.

Let's find $\frac{d^2y}{dx^2}$

As $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So, lets first find dy/dx using parametric form and differentiate it again.

$\frac{dy}{d\theta} = \frac{d}{d\theta}a\,(\theta + \sin\theta) = a(1 + \cos\theta)$ …..equation 3

Similarly,

$\frac{dx}{d\theta} = \frac{d}{d\theta}a\,(1 - \cos\theta) = a\sin\theta$ ……equation 4

$[\because \frac{d}{dx}\cos x = -\sin x, \frac{d}{dx}\sin x = \cos x]$

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a(1+\cos\theta)}{a\sin\theta} = \frac{(1+\cos\theta)}{\sin\theta}$ …..equation 5

Differentiating again w.r.t x :

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{(1+\cos\theta)}{\sin\theta}\right) = \frac{d}{dx}(1+\cos\theta)\operatorname{cosec}\theta$$

Using product rule and chain rule of differentiation together:

$$\frac{d^2y}{dx^2} = \{\operatorname{cosec}\theta\frac{d}{d\theta}(1+\cos\theta) + (1+\cos\theta)\frac{d}{d\theta}\operatorname{cosec}\theta\}\frac{d\theta}{dx}$$

$\frac{d^2y}{dx^2} = \{\operatorname{cosec}\theta(-\sin\theta) + (1+\cos\theta)(-\operatorname{cosec}\theta\cot\theta)\}\frac{1}{a\sin\theta}$ [using equation 4]

$$\frac{d^2y}{dx^2} = \{-1 - \operatorname{cosec}\theta\cot\theta - \cot^2\theta\}\frac{1}{a\sin\theta}$$

As we have to find $\frac{d^2y}{dx^2} = -\frac{1}{a}$ at $\theta = \frac{\pi}{2}$

∴ put θ = π/2 in above equation:

$$\frac{d^2y}{dx^2} = \{-1 - \text{cosec}\frac{\pi}{2}\cot\frac{\pi}{2} - \cot^2\frac{\pi}{2}\}\frac{1}{a\sin\frac{\pi}{2}} = \frac{\{-1-0-0\}1}{a}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{a} \text{ ....ans}$$

16. Question

If x = a (1 + cos θ), y = a (θ+ sinθ) Prove that $\frac{d^2y}{dx^2} = \frac{-1}{a}$ at $\theta = \frac{\pi}{2}$.

Answer

Idea of parametric form of differentiation:

If y = f (θ) and x = g(θ) i.e. y is a function of θ and x is also some other function of θ.

Then dy/dθ = f’(θ) and dx/dθ = g’(θ)

We can write : $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

Given,

y = a (θ + sin θ) ......equation 1

x = a (1+ cos θ) ......equation 2

to prove : $\frac{d^2y}{dx^2} = -\frac{1}{a}$ at $\theta = \frac{\pi}{2}$.

We notice a second-order derivative in the expression to be proved so first take the step to find the second order derivative.

Let’s find $\frac{d^2y}{dx^2}$

As, $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So, lets first find dy/dx using parametric form and differentiate it again.

$\frac{dy}{d\theta} = \frac{d}{d\theta}a\,(\theta + \sin\theta) = a(1 + \cos\theta)$ .....equation 3

Similarly,

$\frac{dx}{d\theta} = \frac{d}{d\theta}a\,(1 + \cos\theta) = -a\sin\theta$ ......equation 4

$[\because \frac{d}{dx}\cos x = -\sin x, \frac{d}{dx}\sin x = \cos x]$

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a(1+\cos\theta)}{-a\sin\theta} = -\frac{(1+\cos\theta)}{\sin\theta}$ .....equation 5

Differentiating again w.r.t x :

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\frac{(1+\cos\theta)}{\sin\theta}\right) = -\frac{d}{dx}(1+\cos\theta)\text{cosec}\,\theta$$

Using product rule and chain rule of differentiation together:

$$\frac{d^2y}{dx^2} = -\{\text{cosec}\,\theta\frac{d}{d\theta}(1+\cos\theta) + (1+\cos\theta)\frac{d}{d\theta}\text{cosec}\,\theta\}\frac{d\theta}{dx}$$

$$\frac{d^2y}{dx^2} = -\{\text{cosec}\,\theta(-\sin\theta) + (1+\cos\theta)(-\text{cosec}\,\theta\cot\theta)\}\frac{1}{(-a\sin\theta)}$$

[using equation 4]

$$\frac{d^2y}{dx^2} = \{-1 - \operatorname{cosec}\theta\cot\theta - \cot^2\theta\}\frac{1}{a\sin\theta}$$

As we have to find $\frac{d^2y}{dx^2} = -\frac{1}{a}$ at $\boldsymbol{\theta} = \frac{\pi}{2}$

∴ put θ = π/2 in above equation:

$$\frac{d^2y}{dx^2} = \{-1 - \operatorname{cosec}\frac{\pi}{2}\cot\frac{\pi}{2} - \cot^2\frac{\pi}{2}\}\frac{1}{a\sin\frac{\pi}{2}} = \frac{\{-1-0-0\}1}{a}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{a}$$

17. Question

If x = cos θ, y = $\sin^3\theta$. Prove that $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 3\sin^2\theta(5\cos^2\theta - 1)$

Answer

The idea of parametric form of differentiation:

If y = f (θ) and x = g(θ), i.e. y is a function of θ and x is also some other function of θ.

Then dy/dθ = f'(θ) and dx/dθ = g'(θ)

We can write : $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

Given,

y = $\sin^3\theta$ ......equation 1

x = cos θ ......equation 2

To prove: $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 3\sin^2\theta\,(5\cos^2\theta - 1)$

We notice a second-order derivative in the expression to be proved so first take the step to find the second order derivative.

Let's find $\frac{d^2y}{dx^2}$

As, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

So, lets first find dy/dx using parametric form and differentiate it again.

$\frac{dx}{d\theta} = -\sin\theta$ ....equation 3

Applying chain rule to differentiate $\sin^3\theta$:

$\frac{dy}{d\theta} = 3\sin^2\theta\cos\theta$ ....equation 4

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3\sin^2\theta\cos\theta}{-\sin\theta} = -3\sin\theta\cos\theta$ ....equation 5

Again differentiating w.r.t x:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(-3\sin\theta\cos\theta)$$

Applying product rule and chain rule to differentiate:

$$\frac{d^2y}{dx^2} = -3\{\sin\theta \frac{d}{d\theta}\cos\theta + \cos\theta \frac{d}{d\theta}\sin\theta\}\frac{d\theta}{dx}$$

$$\frac{d^2y}{dx^2} = 3\{-\sin^2\theta + \cos^2\theta\}\frac{1}{\sin\theta}$$

[using equation 3 to put the value of dθ/dx]

Multiplying y both sides to approach towards the expression we want to prove-

$$y\frac{d^2y}{dx^2} = 3\{-\sin^2\theta + \cos^2\theta\}\frac{y}{\sin\theta}$$

$$y\frac{d^2y}{dx^2} = 3\{-\sin^2\theta + \cos^2\theta\}\sin^2\theta$$

[from equation 1, substituting for y]

Adding equation 5 after squaring it:

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 3\{-\sin^2\theta + \cos^2\theta\}\sin^2\theta + 9\sin^2\theta\cos^2\theta$$

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 3\sin^2\theta\{-\sin^2\theta + \cos^2\theta + 3\cos^2\theta\}$$

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 3\sin^2\theta\{5\cos^2\theta - 1\} \text{ ... . proved}$$

18. Question

If y = sin (sin x), prove that : $\frac{d^2y}{dx^2} + \tan x.\frac{dy}{dx} + y\cos^2 x = 0$

Answer

Given,

y = sin (sin x) ......equation 1

To prove: $\frac{d^2y}{dx^2} + \tan x.\frac{dy}{dx} + y\cos^2 x = 0$

We notice a second-order derivative in the expression to be proved so first take the step to find the second order derivative.

Let's find $\frac{d^2y}{dx^2}$

As $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So, lets first find dy/dx

$$\frac{dy}{dx} = \frac{d}{dx}\sin(\sin x)$$

Using chain rule, we will differentiate the above expression

Let t = sin x $\Longrightarrow \frac{dt}{dx} = \cos x$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$$

$\frac{dy}{dx} = \cos t \cos x = \cos(\sin x) \cos x$ ....equation 2

Again differentiating with respect to x applying product rule:

$$\frac{d^2y}{dx^2} = \cos x \frac{d}{dx}\cos(\sin x) + \cos(\sin x)\frac{d}{dx}\cos x$$

Using chain rule again in the next step-

$$\frac{d^2y}{dx^2} = -\cos x \cos x \sin(\sin x) - \sin x \cos(\sin x)$$

$$\frac{d^2y}{dx^2} = -y\cos^2 x - \tan x \cos x \cos(\sin x)$$

[using equation 1 : y =sin (sin x)]

And using equation 2, we have:

$$\frac{d^2y}{dx^2} = -y\cos^2 x - \tan x \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} + y\cos^2 x + \tan x \frac{dy}{dx} = 0 \ \ldots\ldots \text{proved}$$

19. Question

If $y = (\sin^{-1} x)^2$, prove that: $(1-x^2)\, y_2 - xy_1 - 2=0$

Answer

Note: y_2 represents second order derivative i.e. $\frac{d^2y}{dx^2}$ and y_1 = dy/dx

Given,

$y = (\sin^{-1} x)^2$ ......equation 1

to prove : $(1-x^2)\, y_2 - xy_1 - 2=0$

We notice a second–order derivative in the expression to be proved so first take the step to find the second order derivative.

Let's find $\frac{d^2y}{dx^2}$

As $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

So, lets first find dy/dx

$$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1} x)^2$$

Using chain rule we will differentiate the above expression

Let $t = \sin^{-1} x \Rightarrow \frac{dt}{dx} = \frac{1}{\sqrt{(1-x^2)}}$ [using formula for derivative of $\sin^{-1}x$]

And $y = t^2$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$$

$\frac{dy}{dx} = 2t\frac{1}{\sqrt{(1-x^2)}} = 2\sin^{-1} x\ \frac{1}{\sqrt{(1-x^2)}}$ ....equation 2

Again differentiating with respect to x applying product rule:

$$\frac{d^2y}{dx^2} = 2\sin^{-1}x\,\frac{d}{dx}\left(\frac{1}{\sqrt{1-x^2}}\right) + \frac{2}{\sqrt{(1-x^2)}}\frac{d}{dx}\sin^{-1}x$$

$$\frac{d^2y}{dx^2} = -\frac{2\sin^{-1}x}{2(1-x^2)\sqrt{1-x^2}}(-2x) + \frac{2}{(1-x^2)}\text{[using } \frac{d}{dx}(x^n) = nx^{n-1}\ \frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{(1-x^2)}}\text{]}$$

$$\frac{d^2y}{dx^2} = \frac{2x\sin^{-1}x}{(1-x^2)\sqrt{1-x^2}} + \frac{2}{(1-x^2)}$$

$$(1-x^2)\frac{d^2y}{dx^2} = 2 - \frac{2x\sin^{-1}x}{\sqrt{1-x^2}}$$

Using equation 2 :

$$(1-x^2)\frac{d^2y}{dx^2} = 2 - x\frac{dy}{dx}$$

∴ $(1-x^2)\ y_2-xy_1-2=0$ ......proved

20. Question

If $y = (\sin^{-1} x)^2$, prove that: $(1-x^2)\ y_2-xy_1-2=0$

Answer

Note: y_2 represents second order derivative i.e. $\frac{d^2y}{dx^2}$ and $y_1 = dy/dx$

Given,

$y = (\sin^{-1} x)^2$ ......equation 1

to prove : $(1-x^2)\ y_2-xy_1-2=0$

We notice a second order derivative in the expression to be proved so first take the step to find the second order derivative.

Let’s find $\frac{d^2y}{dx^2}$

As, $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So, lets first find dy/dx

$$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}x)^2$$

Using chain rule we will differentiate the above expression

Let $t = \sin^{-1} x$ => $\frac{dt}{dx} = \frac{1}{\sqrt{(1-x^2)}}$ [using formula for derivative of $\sin^{-1}x$]

And $y = t^2$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$$

$\frac{dy}{dx} = 2t\frac{1}{\sqrt{(1-x^2)}} = 2\sin^{-1}x\ \frac{1}{\sqrt{(1-x^2)}}$ ....equation 2

Again differentiating with respect to x applying product rule:

$$\frac{d^2y}{dx^2} = 2\sin^{-1}x\,\frac{d}{dx}\left(\frac{1}{\sqrt{1-x^2}}\right) + \frac{2}{\sqrt{(1-x^2)}}\frac{d}{dx}\sin^{-1}x$$

$$\frac{d^2y}{dx^2} = -\frac{2\sin^{-1}x}{2(1-x^2)\sqrt{1-x^2}}(-2x) + \frac{2}{(1-x^2)}\text{[using } \frac{d}{dx}(x^n) = nx^{n-1}\ \frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{(1-x^2)}}\text{]}$$

$$\frac{d^2y}{dx^2} = \frac{2x\sin^{-1}x}{(1-x^2)\sqrt{1-x^2}} + \frac{2}{(1-x^2)}$$

$$(1-x^2)\frac{d^2y}{dx^2} = 2 + \frac{2x\sin^{-1}x}{\sqrt{1-x^2}}$$

Using equation 2 :

$$(1-x^2)\frac{d^2y}{dx^2} = 2 + x\frac{dy}{dx}$$

∴ $(1-x^2)\, y_2 - xy_1 - 2 = 0$ ......proved

21. Question

If y $= e^{\tan-1x}$, Prove that: $(1+x^2)y_2+(2x-1)y_1=0$

Answer

Note: y_2 represents second order derivative i.e. $\frac{d^2y}{dx^2}$ and y_1 = dy/dx

Given,

$y = e^{\tan-1x}$ ......equation 1

to prove : $(1+x^2)y_2+(2x-1)y_1=0$

We notice a second order derivative in the expression to be proved so first take the step to find the second order derivative.

Let's find $\frac{d^2y}{dx^2}$

As, $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So, lets first find dy/dx

$$\frac{dy}{dx} = \frac{d}{dx}e^{\tan^{-1}x}$$

Using chain rule we will differentiate the above expression

Let t = $\tan^{-1}$ x => $\frac{dt}{dx} = \frac{1}{1+x^2}$ $[\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}]$

And y = e^t

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$$

$\frac{dy}{dx} = e^t\frac{1}{1+x^2} = \frac{e^{\tan^{-1}x}}{1+x^2}$.......equation 2

Again differentiating with respect to x applying product rule:

$$\frac{d^2y}{dx^2} = e^{\tan^{-1}x}\frac{d}{dx}\left(\frac{1}{1+x^2}\right) + \frac{1}{1+x^2}\frac{d}{dx}e^{\tan^{-1}x}$$

Using chain rule we will differentiate the above expression-

$\frac{d^2y}{dx^2} = \left(\frac{e^{\tan^{-1}x}}{(1+x^2)^2}\right) - \frac{2xe^{\tan^{-1}x}}{(1+x^2)^2}$ [using equation 2 ; $\frac{d}{dx}(x^n) = nx^{n-1}$ & $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$]

$$(1+x^2)\frac{d^2y}{dx^2} = \frac{e^{\tan^{-1}x}}{1+x^2} - \frac{2xe^{\tan^{-1}x}}{1+x^2}$$

$$(1+x^2)\frac{d^2y}{dx^2} = \frac{e^{\tan^{-1}x}}{1+x^2}(1-2x)$$

Using equation 2 :

$$(1+x^2)\frac{d^2y}{dx^2} = \frac{dy}{dx}(1-2x)$$

∴ $(1+x^2)y_2+(2x-1)y_1=0$ ......proved

22. Question

If y = 3 cos (log x) + 4 sin (log x), prove that: $x^2y_2+xy_1+y=0$.

Answer

Note: y_2 represents second order derivative i.e. $\frac{d^2y}{dx^2}$ and $y_1 = dy/dx$

Given,

y = 3 cos (log x) + 4 sin (log x) ......equation 1

to prove: $x^2y_2+xy_1+y=0$

We notice a second order derivative in the expression to be proved so first take the step to find the second order derivative.

Let's find $\frac{d^2y}{dx^2}$

As, $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So, lets first find dy/dx

$$\frac{dy}{dx} = \frac{d}{dx}(3\cos(\log x) + 4\sin(\log x))$$

Let, log x = t

∴ y = 3cos t + 4sin t ....equation 2

$$\frac{dy}{dt} = -3\sin t + 4\cos t$$

$\frac{dt}{dx} = \frac{1}{x}$ ....equation 3

$$\therefore \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$$

$\frac{dy}{dx} = (-3\sin t + 4\cos t)\frac{1}{x}$ ....equation 4

Again differentiating w.r.t x:

Using product rule of differentiation we have

$$\frac{d^2y}{dx^2} = (-3\sin t + 4\cos t)\frac{d}{dx}\frac{1}{x} + \frac{1}{x}\frac{d}{dx}(-3\sin t + 4\cos t)$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}(-3\sin t + 4\cos t) + \frac{1}{x}\frac{dt}{dx}(-3\cos t - 4\sin t)$$

Using equation 2,3 and 4 we can substitute above equation as:

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}x\frac{dy}{dx} + \frac{1}{x}\frac{1}{x}(-y)$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2}$$

Multiplying x^2 both sides:

$$x^2\frac{d^2y}{dx^2} = -x\frac{dy}{dx} - y$$

$\therefore x^2y_2+xy_1+ y =0$ ....proved

23. Question

If $y=e^{2x}(ax + b)$, show that $y_2-4y_1+4y = 0$.

Answer

Note: y_2 represents second order derivative i.e. $\frac{d^2y}{dx^2}$ and $y_1 = dy/dx$

Given,

$y = e^{2x}(ax + b)$ ......equation 1

to prove: $y_2-4y_1+4y = 0$

We notice a second order derivative in the expression to be proved so first take the step to find the second order derivative.

Let's find $\frac{d^2y}{dx^2}$

As, $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So, lets first find dy/dx

$\because y = e^{2x}(ax + b)$

Using product rule to find dy/dx:

$$\frac{dy}{dx} = e^{2x}\frac{dy}{dx}(ax+b) + (ax+b)\frac{d}{dx}e^{2x}$$

$$\frac{dy}{dx} = ae^{2x} + 2(ax+b)e^{2x}$$

$\frac{dy}{dx} = e^{2x}(a + 2ax + 2b)$ ....equation 2

Again differentiating w.r.t x using product rule:

$$\frac{d^2y}{dx^2} = e^{2x}\frac{dy}{dx}(a + 2ax + 2b) + (a + 2ax + 2b)\frac{d}{dx}e^{2x}$$

$\frac{d^2y}{dx^2} = 2ae^{2x} + 2(a + 2ax + 2b)e^{2x}$.......equation 3

In order to prove the expression try to get the required form:

Subtracting 4*equation 2 from equation 3:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} = 2ae^{2x} + 2(a + 2ax + 2b)e^{2x} - 4e^{2x}(a + 2ax + 2b)$$

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} = 2ae^{2x} - 2e^{2x}(a + 2ax + 2b)$$

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} = -4e^{2x}(ax + b)$$

Using equation 1:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} = -4y$$

∴ $y_2 - 4y_1 + 4y = 0$ ……..proved

24. Question

If $x = \sin\left(\frac{1}{a}\log y\right)$, show that $(1-x^2)y_2 - xy_1 - a^2 y = 0$

Answer

Note: y_2 represents second order derivative i.e. $\frac{d^2y}{dx^2}$ and $y_1 = dy/dx$

Given,

$x = \sin\left(\frac{1}{a}\log y\right)$

$(\log y) = a\sin^{-1}x$

$y = e^{a\sin^{-1}x}$ ……equation 1

to prove: $(1-x^2)y_2 - xy_1 - a^2 y = 0$

We notice a second order derivative in the expression to be proved so first take the step to find the second order derivative.

Let's find $\frac{d^2y}{dx^2}$

As, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

So, lets first find dy/dx

∵ $y = e^{a\sin^{-1}x}$

Let $t = a\sin^{-1}x \Rightarrow \frac{dt}{dx} = \frac{a}{\sqrt{(1-x^2)}}$ $\left[\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{(1-x^2)}}\right]$

And $y = e^t$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$$

$\frac{dy}{dx} = e^t\frac{a}{\sqrt{(1-x^2)}} = \frac{ae^{a\sin^{-1}x}}{\sqrt{(1-x^2)}}$ …….equation 2

Again differentiating with respect to x applying product rule:

$$\frac{d^2y}{dx^2} = ae^{a\sin^{-1}x}\frac{d}{dx}\left(\frac{1}{\sqrt{1-x^2}}\right) + \frac{a}{\sqrt{(1-x^2)}}\frac{d}{dx}e^{a\sin^{-1}x}$$

Using chain rule and equation 2:

$\frac{d^2y}{dx^2} = -\frac{ae^{a\sin^{-1}x}}{2(1-x^2)\sqrt{1-x^2}}(-2x) + \frac{a^2e^{a\sin^{-1}x}}{(1-x^2)}$ [using $\frac{d}{dx}(x^n) = nx^{n-1}$ $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{(1-x^2)}}$]

$$\frac{d^2y}{dx^2} = \frac{xae^{a\sin^{-1}x}}{(1-x^2)\sqrt{1-x^2}} + \frac{a^2e^{a\sin^{-1}x}}{(1-x^2)}$$

$$(1-x^2)\frac{d^2y}{dx^2} = a^2e^{a\sin^{-1}x} + \frac{xae^{a\sin^{-1}x}}{\sqrt{1-x^2}}$$

Using equation 1 and equation 2 :

$$(1-x^2)\frac{d^2y}{dx^2} = a^2y + x\frac{dy}{dx}$$

∴ $(1-x^2)y_2 - xy_1 - a^2y = 0$......proved

25. Question

If log y = $\tan^{-1}$ X, show that : $(1+x^2)y_2+(2x-1)\,y_1=0$.

Answer

Note: y_2 represents second order derivative i.e. $\frac{d^2y}{dx^2}$ and y_1 = dy/dx

Given,

log y = $\tan^{-1}$ X

∴ y = $e^{\tan^{-1}x}$ ......equation 1

to prove : $(1+x^2)y_2+(2x-1)y_1=0$

We notice a second order derivative in the expression to be proved so first take the step to find the second order derivative.

Let's find $\frac{d^2y}{dx^2}$

As $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$

So, lets first find dy/dx

$$\frac{dy}{dx} = \frac{d}{dx}e^{\tan^{-1}x}$$

Using chain rule, we will differentiate the above expression

Let t = $\tan^{-1}$ x => $\frac{dt}{dx} = \frac{1}{1+x^2}$ $[\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}]$

And y = e^t

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$$

$\frac{dy}{dx} = e^t\frac{1}{1+x^2} = \frac{e^{\tan^{-1}x}}{1+x^2}$.......equation 2

Again differentiating with respect to x applying product rule:

$$\frac{d^2y}{dx^2} = e^{\tan^{-1}x}\frac{d}{dx}\left(\frac{1}{1+x^2}\right) + \frac{1}{1+x^2}\frac{d}{dx}e^{\tan^{-1}x}$$

Using chain rule we will differentiate the above expression-

$$\frac{d^2y}{dx^2} = \left(\frac{e^{\tan^{-1}x}}{(1+x^2)^2}\right) - \frac{2xe^{\tan^{-1}x}}{(1+x^2)^2}$$

[using equation 2 ; $\frac{d}{dx}(x^n) = nx^{n-1}$ & $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$]

$$(1+x^2)\frac{d^2y}{dx^2} = \frac{e^{\tan^{-1}x}}{1+x^2} - \frac{2xe^{\tan^{-1}x}}{1+x^2}$$

$$(1+x^2)\frac{d^2y}{dx^2} = \frac{e^{\tan^{-1}x}}{1+x^2}(1-2x)$$

Using equation 2 :

$$(1+x^2)\frac{d^2y}{dx^2} = \frac{dy}{dx}(1-2x)$$

∴ $(1+x^2)y_2+(2x-1)y_1=0$ ......proved

MCQ

1. Question

Write the correct alternative in the following:

If x = a cos nt – b sin nt, then $\frac{d^2x}{dt^2}$ is

A. n^2x

B. $-n^2x$

C. –nx

D. nx

Answer

Given:

x=a cos nt-b sin nt

$$\frac{dx}{dt} = -an\ \sin nt - bn \cos nt$$

$$\frac{d^2x}{dt^2} = -an^2 \cos nt + bn^2 \sin nt$$

$= -n^2$ (a cos nt-b sin nt)

$= -\ n^2\ x$

2. Question

Write the correct alternative in the following:

If $x = at^2$, y = 2at, then $\frac{d^2y}{dx^2} =$

A. $-\frac{1}{t^2}$

B. $\frac{1}{2at^3}$

C. $-\frac{1}{t^3}$

D. $-\frac{1}{2at^3}$

Answer

Given:

y = 2at, $x = at^2$

$$\frac{dx}{dt} = 2at\,;\ \frac{dy}{dt} = 2a$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{1}{t}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$= \frac{\frac{-1}{t^2}}{2at}$$

$$= \frac{-1}{2at^3}$$

3. Question

Write the correct alternative in the following:

If y = ax^{n+1} + b x^{-n}, then $x^2\frac{d^2y}{dx^2} =$

A. n(n – 1)y

B. n(n + 1)y

C. ny

D. n^2y

Answer

Given:

$$y = ax^{n+1} + bx^{-n}$$

$$\frac{dy}{dx} = (n+1)ax^n + (-n)bx^{-n-1}$$

$$\frac{d^2y}{dx^2} = n(n+1)ax^{n-1} + (-n)(-n-1)bx^{-n-2}$$

$$x^2\frac{d^2y}{dx^2} = x^2\{n(n+1)ax^{n-1} + (-n)(-n-1)bx^{-n-2}\}$$

$$= n(n+1)a\,x^{n-1+2} + n(n+1)bx^{-n-2+2}$$

$=n(n+1)[a\ x^{n+1} + bx^{-n}]$

=n(n+1)y

4. Question

Write the correct alternative in the following:

$$\frac{d^{20}}{dx^{20}}(2\cos x\cos 3x) =$$

A. $2^{20}(\cos 2x - 2^{20}\cos 4x)$

B. $2^{20}(\cos 2x + 2^{20} \cos 4x)$

C. $2^{20}(\sin 2x - 2^{20} \sin 4x)$

D. $2^{20}(\sin 2x - 2^{20} \sin 4x)$

Answer

Given:

Let y=2 cos x cos 3x

$$2\cos A\cos B = \cos\left(\frac{A+B}{2}\right) + \cos\left(\frac{A-B}{2}\right)$$

So y=cos 2x+cos 4x

$$\frac{dy}{dx} = -2\sin 2x - 4\sin 4x$$

$=(-2)^1$ (sin 2x+2^1 sin 4x)

$$\frac{d^2y}{dx^2} = -4\cos 2x - 16\cos 4x$$

$=(-2)^2$ (cos 2x+2^2 cos 4x)

$$\frac{d^3y}{dx^3} = 8\sin 2x + 64\sin 4x$$

$=(-2)^3$ (cos 2x+2^3 cos 4x)

$$\frac{d^4y}{dx^4} = 16\cos 2x + 256\cos 4x$$

$=(-2)^4$ (cos 2x+2^4 cos 4x)

For every odd degree; differential = $=(-2)^n$ (cos 2x+2^n cos 4x);n={1,3,5...}

For every even degree; differential $=(-2)^n$ (cos 2x+2^n cos 4x);n={0,2,4...}

So, $\frac{d^{20}y}{dx^{20}} = (-2)^{20}(\cos 2x + 2^{20}\cos 4x)$

$=(-2)^{20}$ (cos 2x+2^{20} cos 4x);

5. Question

Write the correct alternative in the following:

If $x = t^2$, $y = t^3$, then $\frac{d^2y}{dx^2} =$

A. $\frac{3}{2}$

B. $\frac{3}{4t}$

C. $\frac{3}{2t}$

D. $\frac{3t}{2}$

Answer

Given:

$x = t^2; y = t^3$

$\frac{dy}{dt} = 3t^2; \frac{dx}{dt} = 2t$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t}{2}$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{3t}{2}}{2t}$

$= \frac{3}{4}$

6. Question

Write the correct alternative in the following:

If y = a + bx^2, a, b arbitrary constants, then

A. $\frac{d^2y}{dx^2} = 2xy$

B. $x\frac{d^2y}{dx^2} = y_1$

C. $x\frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 0$

D. $x\frac{d^2y}{dx^2} = 2xy$

Answer

Given:

$y = a + bx^2$

$\frac{dy}{dx} = 2bx$

$\frac{d^2y}{dx^2} = 2b \neq 2xy$

$x\frac{d^2y}{dx^2} = 2bx$

$= \frac{dy}{dx}$

$x\frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 2bx - 2bx + y$

$= y$

7. Question

Write the correct alternative in the following:

If f(x) = (cos x + i sinx) (cos 2x + i sin 2x) (cos 3x + i sin 3x) .... (cos nx + i sin nx) and f(1) = 1, then f'' (1) is equal to

A. $\frac{n(n+1)}{2}$

B. $\left\{\frac{n(n+1)}{2}\right\}^2$

C. $-\left\{\frac{n(n+1)}{2}\right\}^2$

D. none of these

Answer

Given:

f(x) = (cos x + i sinx) (cos 2x + i sin 2x) (cos 3x + i sin 3x) .... (cos nx + i sin nx)

Since $e^{ix} = \cos x + i \sin x$

So, $f(x) = e^{ix} \times e^{i2x} \times e^{i3x} \times e^{i4x} \times \ldots \times e^{inx}$

$f(x) = e^{ix(1+2+3+4+\cdots+n)}$

$= e^{ix\frac{n(n+1)}{2}}$

$f(1) = e^{\frac{in(n+1)}{2}}$

$f'(x) = ix\frac{n(n+1)}{2}e^{ix\frac{n(n+1)}{2}}$

$f''(x) = i^2x^2\left(\frac{n(n+1)}{2}\right)^2 e^{ix\frac{n(n+1)}{2}}$

$f''(x) = -x^2\left(\frac{n(n+1)}{2}\right)^2 e^{ix\frac{n(n+1)}{2}}$

$f''(1) = -1^2\left(\frac{n(n+1)}{2}\right)^2 \times 1$

$= -\left(\frac{n(n+1)}{2}\right)^2$

8. Question

Write the correct alternative in the following:

If y = a sin mx + b cos mx, then $\frac{d^2y}{dx^2}$ is equal to

A. $-m^2y$

B. m^2y

C. –my

D. my

Answer

Given:

$y = a\sin mx + b\cos mx$

$\frac{dy}{dx} = ma\cos mx - mb\sin mx$

$\frac{d^2y}{dx^2} = -m^2 a\sin mx - m^2 b\cos mx$

$= -m^2[a\sin mx + b\cos mx]$

$= -m^2 y$

9. Question

Write the correct alternative in the following:

If $f(x) = \frac{\sin^{-1}x}{\sqrt{(1-x^2)}}$ then $(1 - x^2)\, f'(x) - xf(x) =$

A. 1

B. -1

C. 0

D. none of these

Answer

Given:

$y = f(x) = \frac{\sin^{-1}x}{\sqrt{(1-x^2)}}$

$\frac{dy}{dx} = \frac{1}{\left(\sqrt{(1-x^2)}\right)^2}\left\{\frac{1}{\sqrt{(1-x^2)}}\sqrt{(1-x^2)} - \sin^{-1}x\frac{(-2x)}{2\sqrt{(1-x^2)}}\right\}$

$= \frac{1}{\left(\sqrt{(1-x^2)}\right)^2}\left\{1 + \frac{x\ \sin^{-1}x}{\sqrt{(1-x^2)}}\right\}$

$= \frac{1 + xy}{(1-x^2)}$

$f'(x) = \frac{1 + xf(x)}{(1-x^2)}$

$(1-x^2)f'(x) = 1 + xf(x)$

$(1-x^2)f'(x) - xf(x) = 1$

10. Question

Write the correct alternative in the following:

If $y = \tan^{-1}\left\{\frac{\log_e(e/x^2)}{\log_e(ex^2)}\right\} + \tan^{-1}\left(\frac{3+2\log_e x}{1-6\log_e x}\right)$, then $\frac{d^2y}{dx^2} =$

A. 2

B. 1

C. 0

D. -1

Answer

Given:

$$y=\tan^{-1}\left\{\frac{\log_e\left(\frac{e}{x^2}\right)}{\log_e(ex^2)}\right\}+\tan^{-1}\left\{\frac{3+2\log_e x}{1-6\log_e x}\right\}$$

$$y=\tan^{-1}\left\{\frac{\log_e e-\log_e x^2}{\log_e e+\log_e x^2}\right\}+\tan^{-1}\left\{\frac{3\log_e e+2\log_e x}{1-3\log_e e\times 2\log_e x}\right\}$$

$$y=\tan^{-1}\left\{\frac{1-\log_e x^2}{1+\log_e x^2}\right\}+\tan^{-1}(3\log_e e)+\tan^{-1}(2\log_e x)$$

$$y=\tan^{-1}\left\{\frac{\log_e e-2\log_e x}{1+\log_e e\times 2\log_e x}\right\}+\tan^{-1}(3\log_e e)+\tan^{-1}(2\log_e x)$$

$$y=\tan^{-1}(\log_e e)-\tan^{-1}(2\log_e x)+\tan^{-1}(3\log_e e)+\tan^{-1}(2\log_e x)$$

$$y=\tan^{-1}(1)+\tan^{-1}(3)$$

$$y=\tan^{-1}\left(\frac{1+3}{1-3}\right)=\tan^{-1}(-2)$$

$$\frac{dy}{dx}=0$$

11. Question

Write the correct alternative in the following:

Let f(x) be a polynomial. Then, the second order derivative of $f(e^x)$ is

A. $f''(e^x)\,e^{2x}+f'(e^x)\,e^x$

B. $f''(e^x)\,e^x+f'(e^x)$

C. $f''(e^x)\,e^{2x}+f''(e^x)\,e^x$

D. $f''(e^x)$

Answer

Given:

$$\frac{d}{dx}\left[\frac{d}{dx}f(e^x)\right]=?$$

Since, $\frac{d}{dx}f(g(x))=f'\big(g(x)\big)g'(x)$

So, $\frac{d}{dx}f(e^x)=f'(e^x)e^x$

Also, $\frac{d}{dx}[f(x)g(x)]=f'(x)g(x)+g'(x)f(x)$

So, $\frac{d}{dx}f'(e^x)e^x=f''(x)e^xe^x+e^xf'(x)$

$=f''(x)e^{2x}+e^xf'(x)$

12. Question

Write the correct alternative in the following:

If $y = a\cos(\log_e x) + b\sin(\log_e x)$, then $x^2 y_2 + xy_1 =$

A. 0

B. y

C. –y

D. none of these

Answer

Given:

$y = a\cos(\log_e x) + b\sin(\log_e x)$

$\frac{dy}{dx} = -a\sin(\log_e x)\frac{1}{x} + b\cos(\log_e x)\frac{1}{x}$

$xy_1 = -a\sin(\log_e x) + b\cos(\log_e x)$

$\frac{d^2y}{dx^2} = -a\cos(\log_e x)\frac{1}{x^2} + \frac{1}{x^2}a\sin(\log_e x) - b\sin(\log_e x)\frac{1}{x^2} + b\cos(\log_e x)\frac{1}{x^2}$

$x^2 y_2 = -a\cos(\log_e x) + a\sin(\log_e x) - b\sin(\log_e x) - b\cos(\log_e x)$

$x^2 y_2 + xy_1 = -a\cos(\log_e x) + a\sin(\log_e x) - b\sin(\log_e x) - b\cos(\log_e x) - a\sin(\log_e x) + b\cos(\log_e x)$

$= -a\sin(\log_e x) - b\cos(\log_e x)$

$= -y$

13. Question

Write the correct alternative in the following:

If x = 2at, y = at^2, where a is a constant, then $\frac{d^2y}{dx^2}$ at $x=\frac{1}{2}$ is

A. 1/2a

B. 1

C. 2a

D. none of these

Answer

Given:

x = 2at, y = at^2

$\frac{dx}{dt} = 2a;\ \frac{dy}{dt} = 2at$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = t$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{1}{2a}$

14. Question

Write the correct alternative in the following:

If x = f(t) and y = g(t), then $\frac{d^2y}{dx^2}$ is equal to

A. $\frac{f'g''-g'f''}{(f')^3}$

B. $\frac{f'g''-g'f''}{(f')^2}$

C. $\frac{g''}{f''}$

D. $\frac{f''g'-g''f'}{(g')^3}$

Answer

Given:

x = f(t) and y = g(t)

$$\frac{dx}{dt} = f'(t);\ \frac{dy}{dt} = g'(t)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$= \frac{1}{f'(t)}\left\{\frac{1}{f'(t)^2}(g''(t)f'(t) - f''(t)g'(t))\right\}$$

$$= \frac{(g''(t)f'(t) - f''(t)g'(t))}{(f'(t))^3}$$

15. Question

Write the correct alternative in the following:

If y = sin (m $\sin^{-1}$ x), then (1 – x^2) y_2 – xy_1 is equal to

A. m^2y

B. my

C. $-m^2y$

D. none of these

Answer

Given:

$$y = \sin(m\sin^{-1}x)$$

$$\frac{dy}{dx} = m\cos(m\sin^{-1}x)\frac{1}{\sqrt{(1-x^2)}}$$

$$x\frac{dy}{dx} = \cos(m\sin^{-1}x)\frac{mx}{\sqrt{(1-x^2)}}$$

$$\frac{d^2y}{dx^2}$$

$$= m\left\{\frac{-m\ \sin(m\ \sin^{-1}x)\sqrt{1-x^2}\ \frac{1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{(1-x^2)}}(-2x)\cos(m\sin^{-1}x)}{\left(\sqrt{(1-x^2)}\right)^2}\right\}$$

$$= \frac{m}{(1-x^2)}\left\{-m\ \sin(m\ \sin^{-1}x) + \frac{x}{\sqrt{(1-x^2}}\cos(m\sin^{-1}x)\ \right\}$$

$(1 - x^2)\ y_2 = m\left\{-m\ \sin(m\ \sin^{-1}x) + \frac{x}{\sqrt{(1-x^2}}\cos(m\sin^{-1}x)\ \right\}$

$$= -m^2\ \sin(m\ \sin^{-1}x) + \frac{mx}{\sqrt{(1-x^2}}\cos(m\sin^{-1}x)$$

$(1 - x^2)\ y_2 - xy_1$

$$= -m^2\ \sin(m\ \sin^{-1}x) + \frac{mx}{\sqrt{(1-x^2}}\cos(m\sin^{-1}x)\ -\ \cos(m\sin^{-1}x)\frac{mx}{\sqrt{(1-x^2)}}$$

$$= -m^2\ \sin(m\ \sin^{-1}x)$$

$$= -m^2y$$

16. Question

Write the correct alternative in the following:

If $y = (\sin^{-1} x)^2$, then $(1 - x^2)\ y_2$ is equal to

A. $xy_1 + 2$

B. $xy_1 - 2$

C. $-xy_1 + 2$

D. none of these

Answer

Given:

$$y = (\sin^{-1}x)^2$$

$$\frac{dy}{dx} = 2\sin^{-1}x\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d^2y}{dx^2} = 2\left\{\left(\frac{1}{\sqrt{1-x^2}}\right)^2 + \sin^{-1}x\frac{\frac{2x}{2\sqrt{1-x^2}}}{\left(\sqrt{1-x^2}\right)^2}\right\}$$

$$= 2\left\{\frac{1}{1-x^2} + \sin^{-1}x\frac{x}{\left(\sqrt{1-x^2}\right)^{3/2}}\right\}$$

$(1 - x^2)\ y_2 = 2\left\{1 + \sin^{-1}x\frac{x}{\sqrt{1-x^2}}\right\}$

$$= 2 + x\left\{2\sin^{-1}x\frac{1}{\sqrt{1-x^2}}\right\}$$

$= 2 + xy_1$

17. Question

Write the correct alternative in the following:

If $y = e^{\tan x}$, then $(\cos^2 x)y_2 =$

A. $(1 - \sin 2x)\ y_1$

B. $-(1 + \sin 2x)\ y_1$

C. $(1 + \sin 2x)\ y_1$

D. none of these

Answer

Given:

$y = e^{\tan x}$

$\frac{dy}{dx} = e^{\tan x}(\sec x)^2$

$\frac{d^2y}{dx^2} = e^{\tan x}(\sec x)^2(\sec x)^2 + e^{\tan x} \times 2\sec x \times \tan x \times \sec x$

$= e^{\tan x}(\sec x)^2[(\sec x)^2 + 2\tan x]$

$(\cos^2 x)y_2 = e^{\tan x}[(\sec x)^2 + 2\tan x]$

$= e^{\tan x}\left[\frac{1 + 2\sin x\cos x}{(\cos x)^2}\right]$

$= e^{\tan x}(\sec x)^2[1 + 2\sin x\cos x]$

$= e^{\tan x}(\sec x)^2[1 + \sin 2x]$

$= [1 + \sin 2x]y_1$

18. Question

Write the correct alternative in the following:

If $y = \frac{2}{\sqrt{a^2 - b^2}}\tan^{-1}\left(\frac{a-b}{a+b}\tan\frac{x}{2}\right)$, $a > b > 0$, then

A. $y_1 = \frac{-1}{a + b\cos x}$

B. $y_2 = \frac{b\sin x}{(a + b\cos x)^2}$

C. $y_1 = \frac{1}{a - b\cos x}$

D. $y_2 = \frac{-b\sin x}{(a - b\cos x)^2}$

Answer

Given:

$y = \frac{2}{\sqrt{(a^2 - b^2)}}\tan^{-1}\left(\frac{a-b}{a+b}\tan\frac{x}{2}\right)$

$$\frac{dy}{dx} = \frac{2}{\sqrt{(a^2-b^2)}}\left(\frac{1}{1+\left(\frac{a-b}{a+b}\tan\frac{x}{2}\right)^2}\right)\left(\frac{a-b}{a+b}\right)\left(\sec\frac{x}{2}\right)^2$$

$$= \frac{2}{\sqrt{(a^2-b^2)}}\left(\frac{(a+b)^2}{(a+b)^2+(a-b)^2\left(\tan\frac{x}{2}\right)^2}\right)\left(\frac{a-b}{a+b}\right)\left(\sec\frac{x}{2}\right)^2$$

$$= \frac{2}{\sqrt{(a^2-b^2)}}\left(\frac{(a+b)}{a^2(1+(\tan x)^2)+b^2(1+(\tan x)^2)+2ab(1-(\tan x)^2)}\right)(a - b)\left(\sec\frac{x}{2}\right)^2$$

$$= 2\left(\frac{1}{a^2\left(1+\left(\tan\frac{x}{2}\right)^2\right)+b^2\left(1+\left(\tan\frac{x}{2}\right)^2\right)+2ab\left(1-\left(\tan\frac{x}{2}\right)^2\right)}\right)\sqrt{(a^2 - b^2)}\left(\sec\frac{x}{2}\right)^2$$

Divide numerator and denominator by $\left(1+\left(\tan\frac{x}{2}\right)^2\right)$;

We get:

$$= 2\left(\frac{1}{a^2+b^2+2ab\left(\frac{1-\left(\tan\frac{x}{2}\right)^2}{1+\left(\tan\frac{x}{2}\right)^2}\right)}\right)\sqrt{(a^2-b^2)}\left(\sec\frac{x}{2}\right)^2\frac{1}{1+\left(\tan\frac{x}{2}\right)^2}$$

$$= 2\left(\frac{1}{a^2+b^2+2ab\cos x}\right)\sqrt{(a^2-b^2)}\left(\sec\frac{x}{2}\right)^2\frac{1}{\left(\sec\frac{x}{2}\right)^2}$$

$$= 2\left(\frac{1}{a^2+b^2+2ab\cos x}\right)\sqrt{(a^2-b^2)}$$

$$\frac{d^2y}{dx^2} = 2\sqrt{(a^2-b^2)}\left(\frac{1}{a^2+b^2+2ab\cos x}\right)^2\{-2ab\sin x\}$$

19. Question

Write the correct alternative in the following:

If $y = \frac{ax+b}{x^2+c}$, then $(2xy_1 + y)y_3 =$

A. $3(xy_2 + y_1)y_2$

B. $3(xy_2 + y_2)y_2$

C. $3(xy_2 + y_1)y_1$

D. none of these

Answer

Given:

$$y = \frac{ax+b}{x^2+c}$$

$$\frac{dy}{dx} = \frac{a(x^2+c) - 2x(ax+b)}{(x^2+c)^2}$$

$$= \frac{-ax^2 - 2bx + ac}{(x^2+c)^2}$$

$$2xy_1 = \frac{-ax^3 - 2bx^2 + acx}{(x^2+c)^2}$$

$$\frac{d^2y}{dx^2} = \frac{(-2ax-2b)(x^2+c)^2 - 2(2x)(x^2+c)(-ax^2-2bx+ac)}{(x^2+c)^4}$$

20. Question

Write the correct alternative in the following:

If $y = \log_e\left(\frac{x}{a+bx}\right)^2$, then $x^3 y_2 =$

A. $(xy_1 - y)^2$

B. $(x + y)^2$

C. $\left(\frac{y - xy_1}{y_1}\right)^2$

D. none of these

Answer

Given:

$$y = \left(\log_e\left(\frac{x}{a+bx}\right)\right)^2$$

$$= 2\log_e\left(\frac{x}{a+bx}\right)$$

$$\frac{dy}{dx} = 2\left(\frac{1}{\frac{x}{a+bx}}\right)\left[\frac{a+bx-bx}{(a+bx)^2}\right]$$

$$= 2\left(\frac{a+bx}{x}\right)\left[\frac{a}{(a+bx)^2}\right]$$

$$= \frac{2a}{x(a+bx)}$$

$$= \frac{2a}{(ax+bx^2)}$$

$$x\frac{dy}{dx} = \frac{2ax}{(ax+bx^2)}$$

$$\frac{d^2y}{dx^2} = 2a\left\{\frac{-(a+2bx)}{(ax+bx^2)^2}\right\}$$

$$= (-a-2bx)\frac{dy}{dx}$$

$$x^3\frac{d^2y}{dx^2} = -x^3(a+2bx)\frac{dy}{dx}$$

21. Question

Write the correct alternative in the following:

If x = f(t) cos t – f’(t) sin t and y = f(t) sin t + f’(t) cos t, then $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 =$

A. f(t) – f’’(t)

B. $\{f(t) - f''(t)\}^2$

C. $\{f(t) + f''(t)\}^2$

D. none of these

Answer

Given:

$x = f(t)\cos t - f'(t)\sin t$

$y = f(t)\sin t + f'(t)\cos t$

$\frac{dx}{dt} = f'(t)\cos t - f(t)\sin t - f''(t)\sin t - f'(t)\cos t$

$= -f(t)\sin t - f''(t)\sin t$

$= -\sin t\,[f(t) + f''(t)]$

$\left(\frac{dx}{dt}\right)^2 = \{-\sin t\,[f(t) + f''(t)]\}^2$

$= (\sin t)^2\{f(t) + f''(t)\}^2$

$\frac{dy}{dt} = f'(t)\sin t + f(t)\cos t + f''(t)\cos t - f'(t)\sin t$

$= f(t)\cos t + f''(t)\cos t$

$= \cos t\,[f(t) + f''(t)]$

$\left(\frac{dy}{dt}\right)^2 = \{\cos t\,[f(t) + f''(t)]\}^2$

$= (\cos t)^2\{f(t) + f''(t)\}^2$

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (\sin t)^2\{f(t) + f''(t)\}^2 + (\cos t)^2\{f(t) + f''(t)\}^2$

$= \{f(t) + f''(t)\}^2$

22. Question

Write the correct alternative in the following:

If $y^{1/n} + y^{-1/n} = 2x$, then $(x^2 - 1)y_2 + xy_1 =$

A. $-n^2y$

B. n^2y

C. 0

D. none of these

Answer

Given:

$y^{1/n} + y^{-1/n} = 2x$

$\frac{1}{n} y^{\frac{1}{n}-1} \frac{dy}{dx} + \frac{-1}{n} y^{\frac{-1}{n}-1} \frac{dy}{dx} = 2$

$\frac{1}{n} \frac{dy}{dx} \left\{ y^{\frac{1}{n}-1} - y^{\frac{-1}{n}-1} \right\} = 2$

23. Question

Write the correct alternative in the following:

If $\frac{d}{dx}\left\{x^n - a_1 x^{n-1} + a_2 x^{n-2} + \ldots + (-1)^n a_n\right\}$

$e^x = x^n e^x$,

Then the value of a_r, $0 < r \le n$, is equal to

A. $\frac{n!}{r!}$

B. $\frac{(n-r)!}{r!}$

C. $\frac{n!}{(n-r)!}$

D. none of these

Answer

Given:

$\frac{d}{dx}\{x^n - a_1 x^{n-1} + a_2 x^{n-2} + \cdots + (-1)^n a_n\} e^x = x^n e^x$

$\frac{d}{dx}\{a_0(-1)^0 x^n + a_1(-1)^1 x^{n-1} + a_2(-1)^2 x^{n-2} + \cdots + (-1)^n a_n\} e^x$

$\frac{d}{dx}(x-1)^n$

$(x-1)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} (-1)^k$

So, at k=r;

$a_r = \binom{n}{r}$

Also, $\binom{n}{r} = \binom{n}{n-r}$

So, $a_r = \binom{n}{n-r}$

24. Question

Write the correct alternative in the following:

If $y = x^{n-1} \log x$, then $x^2 y_2 + (3 - 2n) x y_1$ is equal to

A. $-(n-1)^2 y$

B. $(n-1)^2 y$

C. – n^2y

D. n^2y

Answer

Given:

$y = x^{n-1} \log x$

$\frac{dy}{dx} = (n-1)x^{n-2} \log x + \frac{1}{x} x^{n-1}$

$= (n-1)x^{n-2} \log x + x^{n-2}$

$= x^{n-2}[(n-1) \log x + 1]$

$xy_1 = x^{n-1}[(n-1) \log x + 1]$

$= (n-1)y + x^{n-1}$

$(3-2n)xy_1 = (3-2n)[(n-1)y + x^{n-1}]$

$= (3n - 3 - 2n^2 + 2n)y + 3x^{n-1} - 2nx^{n-1}$ (1)

$\frac{d^2y}{dx^2} = (n-1)(n-2)x^{n-3} \log x + \frac{1}{x}(n-1)x^{n-2} + (n-2)x^{n-3}$

$= (n-1)(n-2)x^{n-3} \log x + (n-1)x^{n-3} + (n-2)x^{n-3}$

$= x^{n-3}[(n-1)(n-2) \log x + (n-1) + (n-2)]$

$x^2 y_2 = x^{n-1}[(n-1)(n-2) \log x + (2n-3)]$

$= (n^2 - 3n + 2)y + 2nx^{n-1} - 3x^{n-1}$ (2)

$x^2 y_2 + (3 - 2n) xy_1$

$= (n^2 - 3n + 2)y + 2nx^{n-1} - 3x^{n-1} + (3n - 3 - 2n^2 + 2n)y + 3x^{n-1} - 2nx^{n-1}$

$= (-n^2 + 2n - 1)y$

$= -(n-1)^2 y$

25. Question

Write the correct alternative in the following:

If $xy - \log_e y = 1$ satisfies the equation $x(yy_2 + y_1^2) - y_2 + \lambda yy_1 = 0$, then $\lambda =$

A. -3

B. 1

C. 3

D. none of these

Answer

Given:

$xy - \log_e y = 1$

$xy = \log_e y + 1$

Differentiate w.r.t. ‘x’ on both sides;

$$y + x\frac{dy}{dx} = \frac{1}{y}\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(\frac{1}{y} - x\right) = y$$

$$\frac{dy}{dx} = \frac{y^2}{(1 - xy)}$$

$$\left(\frac{dy}{dx}\right)^2 = \left[\frac{y^2}{(1 - xy)}\right]^2$$

$$= \frac{y^4}{(1 - xy)^2}$$

$$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{y^2}{(1 - xy)}\right]$$

$$= \frac{1}{(1 - xy)^2}\left\{2y\frac{dy}{dx}(1 - xy) - y^2\left(-y + x\frac{dy}{dx}\right)\right\}$$

$$= \frac{1}{(1 - xy)^2}\left\{2y\frac{dy}{dx}(1 - xy) - y^2\left(-y + x\frac{dy}{dx}\right)\right\}$$

$$= \frac{1}{(1 - xy)^2}\left\{2y\frac{dy}{dx}\frac{y^2}{\frac{dy}{dx}} + y^3 + xy^2\frac{dy}{dx}\right\}$$

$$= \frac{1}{(1 - xy)^2}\left\{2y^3 + y^3 + xy^2\frac{dy}{dx}\right\}$$

$$= \frac{1}{(1 - xy)^2}\left\{3y^3 + xy^2\frac{dy}{dx}\right\}$$

$$= \frac{y^2}{(1 - xy)^2}\left\{3y + x\frac{dy}{dx}\right\}$$

$$y\frac{d^2y}{dy^2} = \frac{y^3}{(1 - xy)^2}\left\{3y + x\frac{dy}{dx}\right\}$$

$$y\frac{d^2y}{dy^2} + \left(\frac{dy}{dx}\right)^2 = \frac{y^3}{(1 - xy)^2}\left\{3y + x\frac{dy}{dx}\right\} + \frac{y^4}{(1 - xy)^2}$$

$$= \frac{y^3}{(1 - xy)^2}\left\{3y + x\frac{dy}{dx} + y\right\}$$

$$= \frac{y^3}{(1 - xy)^2}\left\{4y + x\frac{dy}{dx}\right\}$$

$$x\left[y\frac{d^2y}{dy^2} + \left(\frac{dy}{dx}\right)^2\right] = \frac{y^3x}{(1 - xy)^2}\left\{4y + x\frac{dy}{dx}\right\}$$

$$x\left[y\frac{d^2y}{dy^2} + \left(\frac{dy}{dx}\right)^2\right] - \frac{d^2y}{dy^2} = \frac{y^3x}{(1 - xy)^2}\left\{4y + x\frac{dy}{dx}\right\} - \frac{y^2}{(1 - xy)^2}\left\{3y + x\frac{dy}{dx}\right\}$$

$$= \frac{y^2}{(1 - xy)^2}\left\{xy\left(4y + x\frac{dy}{dx}\right) - 3y - x\frac{dy}{dx}\right\}$$

$$= \frac{y^2}{(1 - xy)^2}\left\{4xy^2 + x^2y\frac{dy}{dx} - 3y - x\frac{dy}{dx}\right\}$$

$$=\frac{y^2}{(1-xy)^2}\left\{y(4xy-3)+x\frac{dy}{dx}(xy-1)\right\}$$

$$=\frac{y^2}{(1-xy)^2}\left\{y(xy+3xy-3)-x\frac{dy}{dx}(1-xy)\right\}$$

$$=\frac{y^2}{(1-xy)^2}\left\{y(xy-3(1-xy))-x\frac{dy}{dx}\frac{y^2}{\frac{dy}{dx}}\right\}$$

$$=\frac{y^2}{(1-xy)^2}\left\{y\left(xy-3\frac{y^2}{\frac{dy}{dx}}\right)-xy^2\right\}$$

$$=\frac{y^2}{(1-xy)^2}\left\{xy^2-3\frac{y^3}{\frac{dy}{dx}}-xy^2\right\}$$

$$=-\frac{y^2}{(1-xy)^2}\left\{3\frac{y^3}{\frac{dy}{dx}}\right\}$$

Since $x\left[y\frac{d^2y}{dy^2}+\left(\frac{dy}{dx}\right)^2\right]-\frac{d^2y}{dy^2}+\lambda y\frac{dy}{dx}=0$

So, $x\left[y\frac{d^2y}{dy^2}+\left(\frac{dy}{dx}\right)^2\right]-\frac{d^2y}{dy^2}=-\lambda y\frac{dy}{dx}$

$$-\lambda y\frac{dy}{dx}=-\frac{y^2}{(1-xy)^2}\left\{3\frac{y^3}{\frac{dy}{dx}}\right\}$$

$$-\lambda y\frac{y^2}{(1-xy)}=-\frac{y^2}{(1-xy)^2}\left\{3\frac{y^3}{\frac{dy}{dx}}\right\}$$

$$\lambda y=\frac{1}{(1-xy)}\left\{3\frac{y^3}{\frac{dy}{dx}}\right\}$$

$$\lambda=\frac{3y^2}{(1-xy)\frac{dy}{dx}}$$

$$\lambda=\frac{3\frac{dy}{dx}}{\frac{dy}{dx}}$$

$\lambda=3$

26. Question

Write the correct alternative in the following:

If $y^2=ax^2+bx+c$, then $y^3\frac{d^2y}{dx^2}$ is

A. a constant

B. a function of x only

C. a function of y only

D. a function of x and y

Answer

Given:

$$y^2 = ax^2 + bx + c$$

$$y = \sqrt{(ax^2 + bx + c)}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{(ax^2 + bx + c)}} \times (2ax + b)$$

$$\frac{d^2y}{dx^2}$$

$$= \frac{1}{2}\left\{\frac{\left(2a \times \sqrt{ax^2 + bx + c}\right) - \left((2ax + b) \times \frac{1}{2\sqrt{(ax^2 + bx + c)}} \times (2ax + b)\right)}{\left(\sqrt{(ax^2 + bx + c)}\right)^2}\right\}$$

$$= \frac{1}{2}\left\{\frac{\frac{4a(ax^2 + bx + c) - (2ax + b)^2}{2\sqrt{(ax^2 + bx + c}}}{\left(\sqrt{(ax^2 + bx + c)}\right)^2}\right\}$$

$$= \frac{1}{2}\left\{\frac{4a^2x^2 + 4abx + 4ac - 4a^2x^2 - b^2 - 4abx}{\left(\sqrt{(ax^2 + bx + c)}\right)^2 \times 2\sqrt{(ax^2 + bx + c}}\right\}$$

$$= \frac{1}{4}\left\{\frac{4ac - b^2}{\left(\sqrt{(ax^2 + bx + c)}\right)^{\frac{3}{2}}}\right\}$$

$$y^3\frac{d^2y}{dx^2} = \frac{1}{4}\left\{\frac{4ac - b^2}{\left(\sqrt{(ax^2 + bx + c)}\right)^3}\right\} \times \left(\sqrt{(ax^2 + bx + c)}\right)^3$$

$$= \frac{4ac - b^2}{4}$$

Hence, y is a constant.

Very short answer

1. Question

If y = a x^{n+1} + bx^{-n} and $x^2\frac{d^2y}{dx^2} = \lambda y$, then write the value of λ.

Answer

Given:

$y = ax^{n+1} + bx^{-n}$

$$\frac{dy}{dx} = (n + 1)ax^n + (-n)bx^{-n-1}$$

$$\frac{d^2y}{dx^2} = n(n + 1)ax^{n-1} + (-n)(-n - 1)bx^{-n-2}$$

$$x^2\frac{d^2y}{dx^2} = x^2\{n(n + 1)ax^{n-1} + (-n)(-n - 1)bx^{-n-2}\} = \lambda y$$

$\lambda y=n(n+1)a\ x^{n-1+2}+n(n+1)bx^{-n-2+2}$

λy=n(n+1)[a x^(n+1)+bx^(-n)]

$\lambda y=n(n+1)$

$\lambda=n(n+1)$

2. Question

If x = a cos nt – b sin nt and $\frac{d^2y}{dt^2}=\lambda x,$ then find the value of λ.

Answer

Given:

y=a cos nt-b sin nt

$$\frac{dy}{dt}= -an\ \sin nt - bn \cos nt$$

$$\frac{d^2y}{dt^2}= -an^2 \cos nt + bn^2 \sin nt = \lambda y$$

$\lambda y= -n^2$ (a cos nt-b sin nt)

$\lambda y= -n^2\ y$

$\lambda= -n^2$

3. Question

If x = t^2 and y = t^3, where a is a constant, then find $\frac{d^2y}{dx^2}$ at $x=\frac{1}{2}$.

Answer

Given:

$x=t^2$;$y=t^3$

$$\frac{dy}{dt}=3t^2;\ \frac{dx}{dt}=2t$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{3t}{2}$$

$$\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}=\frac{\frac{3t}{2}}{2t}$$

$$=\frac{3}{4}$$

4. Question

If x = 2at, y = at^2, where a is a constant, then find $\frac{d^2y}{dx^2}$ at $x=\frac{1}{2}$.

Answer

Given:

$x = 2at,\ y = at^2$

$$\frac{dx}{dt} = 2a;\ \frac{dy}{dt} = 2at$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$=t$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$= \frac{1}{2a}$$

5. Question

If x = f(t) and y = g(t), then write the value of $\frac{d^2y}{dx^2}$.

Answer

Given:

x = f(t) and y = g(t)

$$\frac{dx}{dt} = f'(t);\ \frac{dy}{dt} = g'(t)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$= \frac{1}{f'(t)}\left\{\frac{1}{f'(t)^2}(g''(t)f'(t) - f''(t)g'(t))\right\}$$

$$= \frac{(g''(t)f'(t) - f''(t)g'(t))}{(f'(t))^3}$$

6. Question

If y = 1 - x + $\frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!}$.... to ∞, then write $\frac{d^2y}{dx^2}$ in terms of y.

Answer

Given:

$$y = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \infty$$

$$\frac{dy}{dx} = 0 - 1 + \frac{2x}{2!} - \frac{3x^2}{3!} - \frac{4x^3}{4!} + \cdots \infty$$

$$\frac{d^2y}{dx^2} = 0 - 0 + 1 - \frac{2x}{2!} + \frac{3x^2}{3!} - \frac{4x^3}{4!} + \cdots \infty$$

$$= 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \infty$$

$$\frac{d^2y}{dx^2} = y$$

7. Question

If y = x + e^x, find $\frac{d^2x}{dy^2}$.

Answer

Given:

$$y = x + e^x$$

$$\frac{d^2x}{d^2y} = \frac{1}{\frac{d^2y}{dx^2}}$$

$$\frac{dy}{dx} = 1 + e^x$$

$$\frac{d^2y}{dx^2} = e^x$$

$$\frac{d^2x}{d^2y} = \frac{1}{e^x}$$

$=e^{-x}$

8. Question

If y = |x – x^2|, then find $\frac{d^2y}{dx^2}$.

Answer

Given:

$$y = |x - x^2|$$

$$y = \begin{cases} x - x^2; x \geq 0 \\ x^2 - x; x \leq 0 \end{cases}$$

$$\frac{dy}{dx} = \begin{cases} 1 - 2x; x \geq 0 \\ 2x - 1; x \leq 0 \end{cases}$$

$$\frac{d^2y}{dx^2} = \begin{cases} -2; x \geq 0 \\ 2; x \leq 0 \end{cases}$$

9. Question

If y = |$\log_e$ x|, find $\frac{d^2y}{dx^2}$.

Answer

Given:

$$y = |\log_e x| \; \forall \, x > 0$$

$$y = \log_e x$$

$$\frac{dy}{dx} = \frac{1}{x} = x^{-1}$$

$$\frac{d^2y}{dx^2} = (-1)x^{-2}$$